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The orbit method and the Virasoro extension of $\mathrm{Diff}^+(S^1)$ I. Orbital integrals

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Abstract

Using the finite dimensional example of $\widetilde{\mathrm{PSU}(1,1)}$, the universal covering of $\mathrm{PSU}(1,1)$, as a guide, we revisit the orbit method as it applies to $\hat{\mathcal{D}}$, the universal central extension of $\mathcal{D} = \mathrm{Diff}^+(S^1)$. We clarify some aspects of the classification of coadjoint orbits, determine boundedness properties of the natural height function on these orbits, and calculate orbital integrals.

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1. Introduction

Let G denote a connected real Lie group. The orbit method of Kirillov, Kostant and others aims to establish a qualitative correspondence between data associated to integral coadjoint orbits of G, on the one hand, and irreducible unitary representations of G, on the other hand. In the forward direction one applies the methods of geometric quantization to produce a representation, and in the reverse direction one computes a momentum map (applied to an orbit of highest weight vectors, if this exists) or a transform of the (properly interpreted) character of a representation, to obtain a coadjoint orbit. These prescriptions are in general ambiguous, as they should be, since the method attempts to relate quantum and classical phenomena (see [7]).

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This paper concerns the orbit method, as it applies to the universal central extension, $\hat{\mathcal{D}}$, of $\mathcal{D} = \mathrm{Diff}^+(S^1)$. This infinite dimensional example has been considered previously by many authors, especially Kirillov (see [6–8] and references), Segal [13–15], and Witten [17].

The plan of the paper is the following. In Section 2 we review the classification of the smooth coadjoint orbits with central charge $c \neq 0$, due essentially to Kirillov and Segal [6,13,14]. For fixed $c > 0$, there is a faithful monodromy map from orbits with central charge c to the space of conjugacy classes of $\widetilde{\mathrm{PSL}(2,\mathbb{R})}$, the universal covering of $\mathrm{PSL}(2,\mathbb{R})$, as depicted in Figs. 1 and 2. Our aim is to generally clarify the classification, in particular by explaining why the monodromy mapping is natural (specifically, exploiting an observation of Kirillov linking supersymmetry and the form of the coadjoint action), and adding some details concerning representatives of orbits.

In Section 3 we consider boundedness properties of $\mathrm{d}/\mathrm{d}\theta$ on these orbits. This question was considered locally in [17], and the corresponding global question was posed there. We find that with the exceptions of a single parabolic orbit, the 'universal Teichmuller orbit', and those orbits below the Teichmuller orbit, as depicted in Fig. 2, $\mathrm{d}/\mathrm{d}\theta$ is unbounded. This leads to an intuitive picture of what the various coadjoint orbits look like, from a Morse-theoretic point of view.

In Section 4 we have recapitulated some of the standard structure theory for the Virasoro algebra, and the orbit correspondence that one would naively expect for highest weight representations. Utilizing the momentum map point of view, the orbit method predicts that unitarizable highest weight representations should correspond to nonparabolic orbits with $\mathrm{d}/\mathrm{d}\theta$ bounded. We use this to complete one part of the argument in Section 3.

In Section 5 we consider orbital integrals. For any Lie group G a coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$ has a canonical symplectic form $\omega_{\mathcal{O}}$. If G is finite dimensional, the corresponding volume element defines a G-invariant measure supported on $\mathcal{O}$, which can often be interpreted as a tempered distribution. One can thus consider the Fourier transform

$$\hat{\mathcal{O}}(x) = \int_{\mathcal{O}} \mathrm{e}^{-\mathrm{i}\langle x,\lambda\rangle} \frac{1}{d!}\omega^d, \quad x \in \mathfrak{g}, \tag{1.1}$$

which is necessarily $\mathrm{Ad}G$-invariant. Often $\hat{\mathcal{O}}$ can be computed exactly because the integrand can be interpreted as an equivariantly closed differential form; in particular when $\mathcal{O}$ is an integral coadjoint orbit, Kirillov and others have proved in many cases that

$$\hat{\mathcal{O}}(x) = j(x)\chi_{\mathcal{O}}(\mathrm{e}^x), \tag{1.2}$$

where

$$j(x) = \det{}^{1/2}\left(\frac{\sinh(\mathrm{ad}(x/2))}{\mathrm{ad}(x/2)}\right) \tag{1.3}$$

and $\chi_{\mathcal{O}}$ is a character of a unitary representation associated to the orbit by geometric quantization (see Section 7.5 of [1,7]).

For an infinite dimensional group the preceding integrals and formulas do not literally make sense. However, in the context of loop groups, Frenkel has given a plausible interpretation of (1.1) and proved a formula of the form (1.2), involving the coadjoint orbits for the universal central extension, the characters of the positive energy representations, and Wiener measure [3]. It is intriguing to note that for $\mathcal{D}$, the j function is essentially the η function.

In Sections 5.1–5.3, there is an extended heuristic discussion of the possible meaning of (1.1). The interpretation we eventually focus on involves conditioned Wiener measure (specifically, 'the Malliavin–Shavgulidze measures'). In Section 5.4 we prove that the naive integrals exist precisely for those orbits which are "below" the critical universal Teichmuller orbit, and they can be evaluated. The form of the answer is somewhat surprising. The integrals are expressed in terms of the power series for the $\hat{A}$-genus and L-genus, which possibly suggests some further link with supersymmetry. The formulas can be applied to obtain a weak form of asymptotic invariance for the Malliavin–Shavgulidze measures.

In part II of this paper, we will elaborate on Kirillov's work on the geometric realization of the unitary highest weight representations (see [18] and [8]).

In Appendix A we have recalled some well-known facts concerning the orbit method as it applies to $\widetilde{\mathrm{PSU}(1,1)}$. This is a fascinating example, which remains mysterious despite its fundamental status. In particular, the interaction between the highest weight, principal and complementary series, for small values of the mass parameter m^2, is especially noteworthy (see Fig. 3, which follows from work of Bargmann).

Notation. Throughout this paper we will view $\mathcal{V} = \mathrm{vect}(S^1)$ as the Lie algebra of $\mathcal{D}$. This means that the bracket of two vector fields is given by

$$\left[f\frac{\mathrm{d}}{\mathrm{d}\theta}, g\frac{\mathrm{d}}{\mathrm{d}\theta}\right] = (f'g - fg')\frac{\mathrm{d}}{\mathrm{d}\theta}, \tag{1.4}$$

which is the opposite of the bracket one obtains by viewing vector fields as differential operators.

The universal covering of $\mathcal{D}$, denoted $\tilde{\mathcal{D}}$, will be identified with

$$\{C^\infty \quad \Psi : \mathbb{R} \to \mathbb{R} : \quad \Psi' > 0, \quad \Psi(t+1) = \Psi(t) + 1\}, \tag{1.5}$$

where

$$\tilde{\mathcal{D}} \to \mathcal{D} : \Psi \to \psi, \quad \psi(\mathrm{e}^{2\pi \mathrm{i} t}) = \mathrm{e}^{2\pi \mathrm{i} \Psi(t)}. \tag{1.6}$$

The real Virasoro algebra, the universal central extension of $\mathcal{V}$, denoted as

$$0 \to \mathbb{R}\kappa \to \hat{\mathcal{V}} \to \mathcal{V} \to 0 \tag{1.7}$$

will be realized using the cocycle

$$c^{\nabla}(v, w) = \frac{1}{2}\int \nabla v \, \mathrm{d}(\nabla w), \tag{1.8}$$

where ∇ denotes the Riemannian connection on S^1, i.e. $\nabla(f(\mathrm{d}/\mathrm{d}t)) = \mathrm{d}f \otimes (\mathrm{d}/\mathrm{d}t)$. One can think of this extension invariantly by viewing c^{∇} as a section of a line bundle over the space of all affine connections on S^1. But for notational simplicity we will prefer to have a fixed coordinate t at all times.

At one point we will consider the oriented 1-manifold I, the unit interval. We denote the corresponding group of automorphisms by $\mathcal{D}_{\mathrm{D}}$ or $\mathrm{Aut}(I)$, and its Lie algebra by $\mathcal{V}_{\mathrm{D}}$ or

$\mathrm{aut}(I)$, which consists of vector fields vanishing at the endpoints, where the D stands for Dirichlet. The Virasoro extension $\hat{\mathcal{V}}_{\mathrm{D}}$ is defined by the same formula as above.

2. Classification of coadjoint orbits with $c \neq 0$

Using our preferred coordinate t for S^1, we have a fixed vector space decomposition $\hat{\mathcal{V}} = \mathcal{V} \oplus \mathbb{R}\kappa$. With respect to this decomposition we can identify the dual of $\hat{\mathcal{V}}$ with the space

$$\hat{\mathcal{V}}^* = \mathbb{R}\kappa^* \oplus \mathcal{V}^*, \tag{2.1}$$

where $\mathcal{V}^*$ is naturally identified with the space of distributional quadratic differentials on S^1.

In this section we are principally interested in the action on the smooth part of the dual, $\mathbb{R}\kappa^* \oplus Q \subset \hat{\mathcal{V}}^*$, where $Q = \{q(\mathrm{d}t)^2 : q \in C^\infty(S^1, \mathbb{R})\}$. The affine spaces $c\kappa^* + Q$, $c \neq 0$, are invariant under the coadjoint action. The basic problem is to determine the orbit structure.

The analysis proceeds as follows. First (following Kirillov and Segal) one observes the nonobvious fact that $c\kappa^* + Q$ is equivariantly isomorphic to a natural but nonstandard action of $\mathcal{D}$ on Hill's operators. The fact that this second action is natural is neatly explained by the existence of a canonical superalgebra extension of $\hat{\mathcal{V}}$ ($N = 1$ superconformal symmetry; for an historical account of the discovery of this structure, see [16]). Secondly, one observes (following Kirillov and Segal) that the space of Hill's equations is equivariantly isomorphic to the space of real projective structures on S^1, hence that the orbits are separated by their monodromy, properly interpreted as conjugacy classes in the universal covering of $\mathrm{PSL}(2, \mathbb{R})$. Finally, one determines the image of the monodromy map, which involves an analysis of stabilizers, initiated by Kirillov [6] and reconsidered by Witten [17]. In the process we find representatives for the orbits, and the corresponding stability subgroups.

In the last section we consider the slight changes necessary in the open string case.

2.1. $N = 1$ *supersymmetry, monodromy and smooth coadjoint orbits for* $c \neq 0$

There is a canonically associated superalgebra extension of $\mathcal{V}$

$$s\mathcal{V} = \Omega^{-1} \oplus \Omega^{-1/2}, \tag{2.2}$$

where $\Omega^{-1} = \mathcal{V}$ has the usual Lie structure, and the commutator map for odd variables

$$\Omega^{-1/2} \otimes \Omega^{-1/2} \to \Omega^{-1} \tag{2.3}$$

is multiplication of half vector fields (where in this paper we understand this square root in terms of the trivial periodic spin structure). Since the map (2.3) is clearly $\mathcal{D}$-equivariant, $s\mathcal{V}$ is indeed a superalgebra. There is also a central extension

$$0 \to \mathbb{R} \to s\hat{\mathcal{V}} \to s\mathcal{V} \to 0. \tag{2.4}$$

As a $\mathbb{Z}_2$-graded vector space

$$s\hat{\mathcal{V}} = \hat{\mathcal{V}} \oplus \Omega^{-1/2}. \tag{2.5}$$

The multiplication of odd variables is given by the symmetric map

$$\Omega^{-1/2} \otimes \Omega^{-1/2} \to \hat{\mathcal{V}} : \Phi, \Psi \to \Phi\Psi - \left(\int \nabla(\Phi)\nabla(\Psi) \right) \kappa, \tag{2.6}$$

note that $\nabla(\Phi)$ is a half form, so that the integral is well-defined. To check that this does indeed form a superalgebra, we need to check the $\mathcal{V}$-equivariance of (2.6). This reduces to checking the identity

$$c^{\nabla}(V, \Phi\Psi) = -\int \left\{ \left(-v\phi' + \frac{1}{2} v'\phi \right)' \psi' + \phi' \left(-v\psi' + \frac{1}{2} v'\psi \right)' \right\}, \tag{2.7}$$

where $V = v(\mathrm{d}/\mathrm{d}t)$, $\Phi = \phi(\mathrm{d}t)^2$, $\Psi = \psi(\mathrm{d}t)^2$. The LHS of (2.7) is $(1/2)\int v'(\phi\psi)''$, and a straightforward calculation shows that this is exactly the RHS.

Dual to the multiplication map (2.7), there is a $\mathcal{D}$-equivariant map

$$\hat{\mathcal{V}}^* \to S^2(\Omega^{-1/2})^*. \tag{2.8}$$

To compute the map suppose that $c\kappa^* + q \in \hat{\mathcal{V}}^*$. The corresponding symmetric bilinear form on $\Omega^{-1/2}$ is given by the composition

$$\begin{aligned} &\Omega^{-1/2} \otimes \Omega^{-1/2} \to \hat{\mathcal{V}} \overset{c\kappa^*+q}{\to} \mathbb{R}, \\ &f\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{1/2} \otimes g\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{1/2} \to fg\frac{\mathrm{d}}{\mathrm{d}t} - \left(\int f'g'\,\mathrm{d}t \right)\kappa \to \int \{-cf'g' + qfg\}\,\mathrm{d}t. \end{aligned} \tag{2.9}$$

If q is an ordinary function, this is the symmetric form that corresponds to the Hill's operator

$$c\kappa^* + q(\mathrm{d}t)^2 : \Omega^{-1/2} \to \Omega^{3/2} \subset (\Omega^{-1/2})^*, \tag{2.10}$$

where $\kappa^*(f(\mathrm{d}/\mathrm{d}t)^{1/2}) = f''(\mathrm{d}/\mathrm{d}t)^{3/2}$. This is Kirillov's explanation of the following lemma (see the cryptic comments in Section 7.6 of [6]).

Lemma 2.1. *The coadjoint action of $\mathcal{D}$ on $c\kappa^* + Q$, $c \neq 0$, is isomorphic to the natural action of $\mathcal{D}$ (by conjugation) on the space of Hill's operators*

$$\{c\kappa^* + q(\mathrm{d}t)^2 : \Omega^{-1/2} \to \Omega^{3/2} : q \in C^\infty(S^1, \mathbb{R})\}.$$

A real projective structure for an oriented n-manifold C, with universal covering $\tilde{C}$, is a pair (M, f), where $M : \mathrm{Aut}(\tilde{C}) \to \mathrm{Aut}(\mathbb{RP}^n)$ is a (monodromy) representation and $f : \tilde{C} \to \mathbb{RP}^n$ is a M-equivariant orientation-preserving immersion. Two such structures are isomorphic if they differ by the natural action of a $g \in \mathrm{Aut}(\mathbb{RP}^n)$ on such structures, $g : (M, f) \to (gMg^{-1}, g \circ f)$. There is a natural action of $\mathrm{Aut}(C)$ on such structures by $\sigma : (M, f) \to (M, f \circ \sigma^{-1})$ which preserves isomorphism classes.

In the one-dimensional case the projection $\mathrm{PSL}(2, \mathbb{R}) \to \mathbb{RP}^1 : g \to g\mathbb{P}(1, 0)$ is a homotopy equivalence. As a consequence, for $C = S^1$ an isomorphism class of projective structures, $[(M, f)]$, determines a conjugacy class in the universal cover $\widetilde{\mathrm{PSL}}(2, \mathbb{R})$ of $\mathrm{PSL}(2, \mathbb{R})$, since f determines a homotopy class of paths from the identity to $M(1)$, hence an element $\widetilde{M(1)}$ covering $M(1)$. Kirillov and Segal independently noted that this leads to a classification of smooth coadjoint orbits for $c \neq 0$.

Proposition 2.1. *There is a $\mathcal{D}$-equivariant map from Hill's operators, as in Lemma 2.1, to the natural action on isomorphism classes of projective structures on S^1, given by*

$$c\kappa^* + q(\mathrm{d}t)^2 \to \mathbb{P}(u_1, u_2),$$

where $u_1, u_2 \in \Omega^{-1/2}$ is a pair of independent solutions of $cu'' + qu = 0$. This map is an isomorphism. The monodromy map

$$c\kappa^* + Q \to [\widetilde{\mathrm{PSL}(2,\mathbb{R})}] : c\kappa^* + q(\mathrm{d}t)^2 \to [\tilde{F}(1)], \tag{2.11}$$

where $\tilde{F}$ is the unique lift of the fundamental solution of $c(\mathrm{d}/\mathrm{d}\theta)^2 + q$, separates the coadjoint orbits.

For notational simplicity, let G and $\tilde{G}$ denote $\mathrm{PSL}(2,\mathbb{R})$ and its universal covering, respectively, viewed as abstract Lie groups. Let *Ell*, *Hyp* and *Par* denote the sets of elliptic, hyperbolic and parabolic elements of G, respectively; the corresponding inverse images will be referred to as the elliptic, hyperbolic and parabolic elements of $\tilde{G}$, respectively.

To work with $\tilde{G}$, it is convenient to identify G with $\mathrm{PSU}(1,1)$, which we view as a subgroup of $\mathcal{D}$. The Poincare rotation number

$$\rho : \tilde{\mathcal{D}} \to \mathbb{R} : \sigma \to \lim_{n\uparrow\infty} \frac{\sigma^n(t)}{n} \tag{2.12}$$

exists, is independent of t, and defines a continuous central function on $\tilde{\mathcal{D}}$. The restriction of ρ to $\tilde{G}$ is characterized by: (i) ρ is integral on nonelliptic elements of $\tilde{G}$; (ii) $\rho(\sigma+n) = \rho(\sigma)+n$, for all $\sigma \in \tilde{G}$, $n \in \mathbb{Z} = C(\tilde{G})$; (iii) $\rho(t \to t+t_0) = t_0$ for $t_0 \in \mathbb{R}$. (*Proof*. Any $g \in G$ has a fixed point as a transformation of the closed disk, by Brouwer's fixed point theorem. If g is nonelliptic, then its fixed points are on the circle. Thus for a nonelliptic $\tilde{g} \in \tilde{G}$, there exists t such that $\tilde{g}$ shifts t by an integer. This leads to (i). Parts (ii) and (iii) are obvious.) This leads to the following standard picture of the conjugacy classes of $\tilde{G}$ and G.

In this picture π denotes the projection from conjugacy classes of $\tilde{G}$ to conjugacy classes of G. The projection $|\mathrm{tr}|$ is the map $G \to \mathbb{R} : g \to |\mathrm{trace}(g)|$ (thus g is elliptic, parabolic, or hyperbolic if and only if $|\mathrm{tr}| < 2, =2, >2$, respectively). The bifurcation point corresponding to the height $\rho = n$ involves three points: $n \in C(\tilde{G})$ (thus 0 corresponds to the identity in $\tilde{G}$), and the conjugacy classes $\mathrm{Par}_n^{\pm}$, represented by the parabolic elements $U^{\pm 1} + n$, respectively, where U is the unique element of $\tilde{G}$ which covers the unipotent element

$$\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{R})$$

and has a fixed point as a transformation of $\mathbb{R}$. As $|\mathrm{tr}| \downarrow 2$, Hyp_n has $\{n\}$ and $\mathrm{Par}_n^{\pm}$ as accumulation points, and the points $\mathrm{Par}_n^{\pm}$ have $\{n\}$ in their closure (so that the topology is not T_1). The image of the exponential map $\exp : \mathfrak{g} \to \tilde{G}$ is the union of Hyp_0 and the closure of the set of elliptic elements (for an alternate picture, see [4]).

In terms of this picture we can now state the fine classification of orbits, essentially due to Kirillov. (Note: In [6], where the statement of the classification first appeared in a very

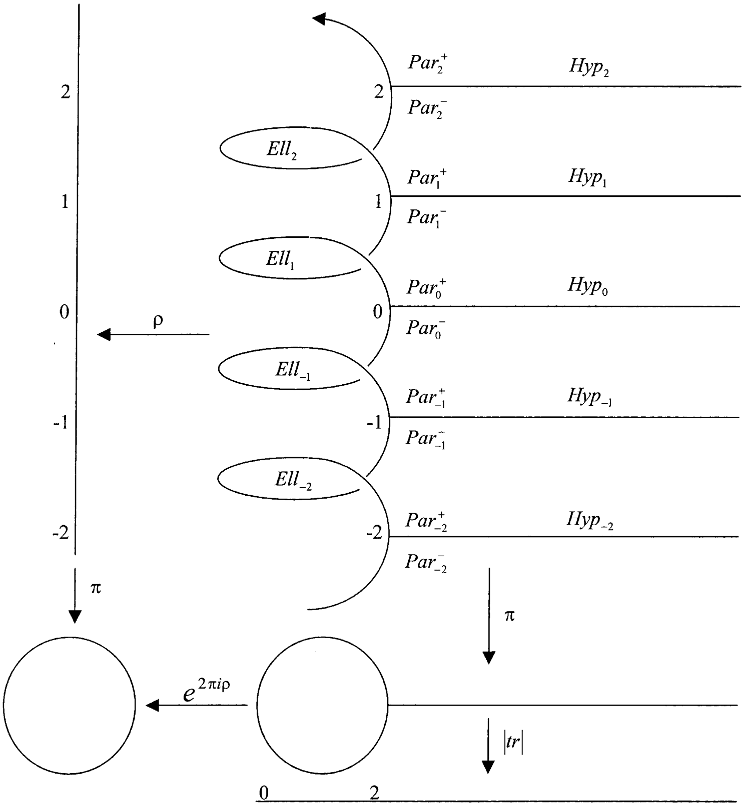

Fig. 1. Conjugacy classes of $\mathrm{PSU}(1,1)$.

nearly correct form, there are several assertions which appear without proof, many of which are completed in [17]. A complete proof, which is quite lengthy, can be found in [2].)

Theorem 2.1. *Fix $c > 0$. The monodromy map (2.11) induces a bijective correspondence between the smooth coadjoint orbits of $\hat{\mathcal{D}}$ with central charge c and the set of conjugacy classes given by $\{\rho \geq 0\} \setminus \{\{0\}, \mathrm{Par}_0^-\}$, in terms of Fig. 1. These coadjoint orbits are represented by the following potentials*:

(1) $q(t) = 0$ *represents* Par_0^+, *with stability subgroup* $\mathrm{Rot}(S^1)$;
(2) $q(t) = ch$, $h < 0$, *represents the orbit of* Hyp_0 *with* $|\mathrm{tr}| = 2\cosh(\sqrt{-h})$, *with stability subgroup* $\mathrm{Rot}(S^1)$;
(3) $q(t) = ch$, $h > 0$, $h \neq (n\pi)^2$ $(n \in \mathbb{Z})$, *represents the orbit of Ell with* $\rho = (1/\pi)\sqrt{h}$, *with stability subgroup* $\mathrm{Rot}(S^1)$;
(4) $q(t) = c(n\pi)^2$, $n > 0$, *represents the orbit* $\{n\} \subset C(\widetilde{\mathrm{PSL}(2,\mathbb{R})})$, *with stability subgroup* $\mathrm{PSU}^{(n)}(1,1) \subset \mathcal{D}$ (*see* Section 4 *below*);
(5) $$q_{(n,\alpha)} = c(n\pi)^2\left(1 + \frac{2\alpha\sin(2\pi nt)(2+\sin(2\pi nt))}{(1+\alpha\sin(2\pi nt))^2}\right), \quad n > 0,$$

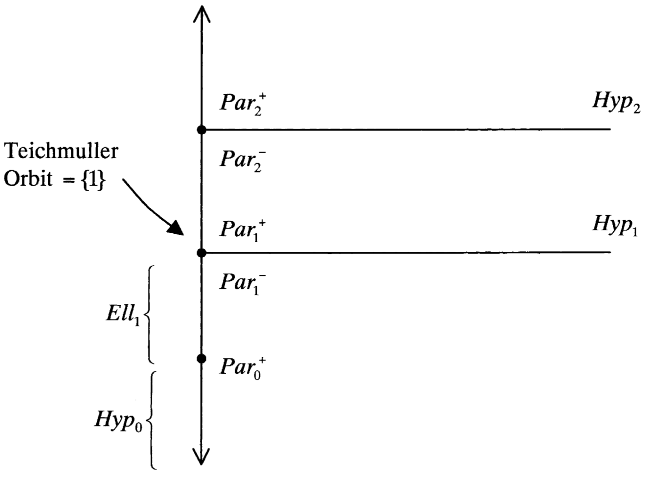

Fig. 2. Smooth coadjoint orbits, $c > 0$.

represents the orbit of Hyp_n *with* $|\mathrm{tr}| = 2\cosh(\alpha/2\sqrt{1-\alpha^2})$, *with stability subgroup* $\exp(\mathbb{R}\xi_{(n,\alpha)}) \times \mathbb{Z}_n$, *where* $\xi_{(n,\alpha)} = \sin(2\pi nt)(1+\alpha\sin(2\pi nt))(\mathrm{d}/\mathrm{d}t)$, $\mathbb{Z}_n$ *is identified with the nth roots of* 1 *in* $\mathrm{Rot}(S^1)$ *and* $0 < \alpha < 1$;

$$q^{\pm}_{(n,\alpha)} = c(n\pi)^2\left(1 \pm \frac{2\alpha(1+\sin(2\pi nt) \pm 2\alpha\cos^2(2\pi nt))}{1\pm\alpha\sin(2\pi nt))^2}\right), \quad n > 0, \tag{6}$$

represents the orbit $\mathrm{Par}^{\pm}_n$, *with stability subgroup* $\exp(\mathbb{R}\xi_{(n,\pm)}) \times \mathbb{Z}_n$, *where* $\xi_{(n,\pm)} = (1+\sin(2\pi nt))(1\pm\alpha\sin(2\pi nt))(\mathrm{d}/\mathrm{d}t)$, $\mathbb{Z}_n$ *is identified with the nth roots of* 1 *in* $\mathrm{Rot}(S^1)$, *for any choice of* α *with* $0 < \alpha < 1$.

It is convenient to think of this parameterization in the following way.

Remark 2.1.

(a) The point of emphasizing this picture, rather than Fig. 1, is that in this paper we cannot detect any significant qualitative differences between the Virasoro orbits corresponding to the classes Hyp_0, Par^+_0 and Ell_1. This is surprising from the $\mathrm{PSL}(2,\mathbb{R})$ point of view.
(b) The most interesting orbit is the one corresponding to $\{1\}$ in (4) of Theorem 2.1 above, which is the first bifurcation point as one ascends Fig. 2. This orbit, together with its symplectic structure, can naturally be identified with (a dense subspace of) Ber's universal Teichmuller space, together with its Weil-Petersson Kahler structure [10]. We will refer to this as the universal Teichmuller orbit.

2.2. *Symmetry group of I*

In this section we note the slight changes necessary to classify the smooth coadjoint orbits for the symmetry group of I, the oriented unit interval, again for central charge $c \neq 0$.

Let $\mathcal{D}_{\mathrm{D}}$ denote the group of orientation-preserving diffeomorphisms of I. The rationale for the notation is that the D stands for Dirichlet, as opposed to the periodic condition for S^1.

Remark 2.2. There is an injection $\tilde{\mathcal{D}}_0 = \mathcal{D}_1 \to \mathcal{D}_{\mathrm{D}}$ (essentially the identity map in terms of the coordinate t). But this is far from an isomorphism because the image consists of diffeomorphisms with relations between the derivatives at the endpoints. One might naively expect that $\mathcal{D}_{\mathrm{D}}$ would be the appropriate symmetry group for open string theory. But in fact, in the present state of string theory, in which a background space–time is always posited, it is $\mathcal{D}_1$ which is the relevant symmetry group.

The Lie algebra, denoted $\mathcal{V}_{\mathrm{D}}$, consists of vector fields along I which vanish on the boundary. There are Virasoro extensions

$$0 \to \mathbb{R}\kappa \to \hat{\mathcal{V}}_{\mathrm{D}} \to \mathcal{V}_{\mathrm{D}} \to 0, \tag{2.13}$$

$$0 \to \mathbb{R}\kappa \to s\hat{\mathcal{V}}_{\mathrm{D}} \to s\mathcal{V}_{\mathrm{D}} \to 0, \tag{2.14}$$

both given by the same formulas as in the periodic case, where $\Omega_{\mathrm{D}}^{-1/2}$ consists of half vector fields that vanish on the boundary.

Remark 2.3. The Virasoro cocycle c^{∇} is in fact a cocycle for the larger algebra $\mathrm{Vect}(I)$; the point is that verification of the cocycle property does not involve an integration by parts, so that the Dirichlet condition on vector fields is irrelevant.

Similarly, there is a superalgebra $s\mathrm{Vect}(I)$. On the other hand, the $N = 1$ superalgebra (2.14) depends upon the Dirichlet condition in an essential way; the point is that the verification of the identity (2.7) does require an integration by parts.

In this case we set $Q_{\mathrm{D}} = \{q(\mathrm{d}t)^2 : q \in C^{\infty}(I)\}$, and the smooth part of the coadjoint action

$$\mathcal{D}_{\mathrm{D}} \times (c\kappa^* + Q_{\mathrm{D}}) \to c\kappa^* + Q_{\mathrm{D}} \tag{2.15}$$

is equivalent to the natural action of $\mathcal{D}_{\mathrm{D}}$ on Hill's operators

$$\{c\kappa^* + q(\mathrm{d}t)^2 : \Omega_{\mathrm{D}}^{-1/2} \to \Omega^{3/2} : q \in C^{\infty}(I)\}, \tag{2.16}$$

provided that $c \neq 0$. This is in turn equivalent to the action on isomorphism classes of projective structures on I, i.e. immersions of I into $\mathbb{RP}^1$, modulo the action of $\mathrm{PSL}(2, \mathbb{R})$.

Let $\sim$ denote the equivalence relation on $\mathbb{R}_{\geq 0}$ corresponding to the partition $0, (0, 1), 1, (1, 2), \ldots$. Given a projective structure on I, the images of 0 and 1 may or may not coincide. If the images coincide, we define the winding number of the structure in the usual way; this number is an integer. If the images do not coincide, then we define the winding number to be the interval $(n, n + 1)$ if n is the largest integer such that the map defining the structure covers $\mathbb{RP}^1$ completely n times.

Proposition 2.2.

(a) *The orbit structure for the action* (2.15) *is determined by the map*

$$\rho_{\mathrm{D}} : c\kappa^* + Q_{\mathrm{D}} \to \mathbb{R}_{\geq 0}/\sim,$$

where a Hill's operator is mapped to the winding number of the corresponding isomorphism class of projective structure.

(b) *A representative for the orbit corresponding to the class of a* $w \geq 0$ *is given by* $q = cw^2$.
(c) *If* $w = n$ *is integral, then* $(\mathcal{D}_{\mathrm{D}})_{q=cn^2} = \mathrm{PSU}(1,1)_1^{(n)}$ (*see* Remark 2.2), *and otherwise the stability subgroup is trivial*.

Proof of Proposition 2.2. It is clear that the winding number is constant on orbits. Given two projective structures with the same winding number, by applying an appropriate $\mathrm{PSL}(2,\mathbb{R})$ transformation, we can suppose that their initial points coincide, and the same for their final points. By considering the map defining the first projective structure, followed by the local inverse to the map defining the second projective structure, we obtain a diffeomorphism of I which relates the two projective structures. [To put this another way, consider the domains of the two maps. Mark off the points that go to the image of the first point (which is the same for both maps). The first subinterval for each map wraps exactly once around $\mathbb{RP}^1$, hence we obtain a diffeomorphism between these two subintervals. We now consider the second subintervals, and so on.] □

3. Boundedness properties of d/dt

We return to the case of S^1. Using the parameterization of coadjoint orbits in $c\kappa^* + Q$ by conjugacy classes, we can view the rotation number as a function on this space of orbits.

Theorem 3.1. *Fix* $c > 0$.

(a) *The function* $\langle \mathrm{d}/\mathrm{d}t, \cdot\rangle$ *is bounded on a* $\mathcal{D}$*-orbit* $\mathcal{O} \subset c\kappa^* + Q$ *if and only if either* $\rho(\mathcal{O}) < 1$, *or* $\rho(\mathcal{O}) = 1$ *and* $\mathcal{O}$ *is either the universal Teichmuller orbit or* Par_1^-.
(b) *An orbit has a critical point if and only if* $\mathcal{O}$ *is represented by a constant potential, in which case the constant potential is the unique critical point*.

Proof of Theorem 3.1. Throughout this proof all integrals will be understood to be over the interval $[0, 1]$. We begin by recalling that the coadjoint action of the Virasoro algebra is given by the formula

$$\mathrm{Ad}^*_{\hat{\mathcal{V}}} : \tilde{\mathcal{D}} \times (\mathbb{R}\kappa^* \oplus Q) \to \mathbb{R}\kappa^* \oplus Q,$$
$$\mathrm{Ad}^*_{\hat{\mathcal{V}}}(\sigma^{-1})(c\kappa^* + q(\mathrm{d}t)^2) = c\kappa^* + \sigma^*(q(\mathrm{d}t)^2) + \tfrac{1}{2}cS(\sigma), \tag{3.1}$$

where $S(\sigma) = (\ln \sigma')'' - \frac{1}{2}(\ln \sigma')'^2$ is the Schwarzian derivative (see [6,13]). Thus

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}t}, \mathrm{Ad}^*_{\hat{\mathcal{V}}}(\sigma^{-1})(c\kappa^* + q(\mathrm{d}t)^2)\right\rangle = \int \left\{q(\sigma)\sigma'^2 + \frac{c}{2}S(\sigma)\right\}\,\mathrm{d}t$$
$$= \int \left\{q(\sigma)\sigma'^2 - \frac{c}{4}b'^2\right\}\,\mathrm{d}t = \int \left\{q(\sigma)\frac{\mathrm{e}^{2b}}{(\int \mathrm{e}^b)^2} - \frac{c}{2}b'^2\right\}\,\mathrm{d}t, \tag{3.2}$$

where $\ln \sigma' - \ln \sigma'(0) = b$.

We now suppose that q is constant. In this case we can write our functional as

$$L(b) = q\frac{\int e^{2b}}{\left(\int e^{b}\right)^2} - \frac{c}{4}\int b'^2, \tag{3.3}$$

where $b \in \mathrm{Path}^{0,0}\mathbb{R}$, $b = \ln \sigma' - \ln \sigma'(0)$. If q is nonpositive, then Holder's inequality implies that $L(b) \le q = L(0)$. Thus the functional L is bounded and the maximum occurs at the constant potential representative for the orbit.

To prove that the same is true for any $q \le c\pi^2$ requires a more sophisticated argument. One method, although quite roundabout, is to use some elementary facts about the unitary highest weight representations. We can suppose that $c \gg 1$. As we show in Section 4.2 below, each of the orbits represented by $q \le c\pi^2$ has a Plucker embedding into the projective space of a unitary lowest weight representation. The basic point is that in this context, classical and quantum energy are identified for elements in the Lie algebra, and it is easier to study the energy operator on the linear representation space than on the curved orbit space. Since the representation is of positive energy type, this implies the same for the corresponding coadjoint orbit. To be precise, in (4.16) take $\hat{\sigma} \cdot v$ in place of v_0 and L_0 in place of x. This implies that the orbit of any $q \le c\pi^2$ is $\mathrm{d}/\mathrm{d}t$-bounded.

Now consider the orbit Par_1^-. Consider a representative q for this orbit as in (6) of Theorem 2.1

$$q = \pi^2\left(1 - 2\alpha\frac{1 + \sin(\theta) - 2\alpha\cos^2(\theta)}{(1 - 2\alpha\sin(\theta))^2}\right), \quad \theta = 2\pi t, \tag{3.4}$$

where we may choose any $0 < \alpha < 1$. If we choose $\alpha < 1/4$, then $1 + \sin(\theta) - 2\alpha\cos^2(\theta) \ge 0$, which implies that $q \le \pi^2$. Therefore (3.3) is bounded by

$$c\pi^2\frac{\int e^{2b}}{\left(\int e^{b}\right)^2} - \frac{c}{4}\int b'^2 \le c\pi^2, \tag{3.5}$$

where we have used the bound for the universal Teichmuller orbit.

We have now shown that $\mathrm{d}/\mathrm{d}t$ is bounded in all cases claimed in (a). We now aim to show that $\mathrm{d}/\mathrm{d}t$ is not bounded for a constant potential $q > c\pi^2$. Suppose that we consider the critical value $q = c\pi^2$. In this case we know that the stability subgroup jumps to $\mathrm{PSU}(1, 1)$. We have been considering $\mathrm{d}/\mathrm{d}t$ as a function on the total space of the bundle

$$\mathrm{PSU}(1,1)/\mathrm{Rot} \to \mathcal{D}/\mathrm{Rot} \to \mathcal{D}/\mathrm{PSU}(1,1). \tag{3.6}$$

The function $\mathrm{d}/\mathrm{d}t$ has the constant value $c\pi^2$ on the entire fiber $\mathrm{PSU}(1, 1)/\mathrm{Rot}$, over the basepoint $q = c\pi^2$. It is easy to calculate this fiber. Consider the hyperbolic element

$$\phi = \begin{pmatrix} c & s \\ s & c \end{pmatrix} \in \mathrm{PSU}(1,1)_1,$$

where $c = \cosh(\beta)$, $s = \sinh(\beta)$. Then $b_\phi(t) = \ln(\Phi'(t)/\Phi'(0))$, where

$$\begin{aligned}\Phi'(t) &= \frac{z\phi_z}{\phi} = \frac{z}{(cz+s)/(sz+c)\cdot(sz+c)^2} = |cz+s|^{-2}\\ &= \frac{1}{c^2+s^2+cs2\cos(2\pi t)} = \frac{1}{\cosh(2\beta)+\sinh(2\beta)\cos(2\pi t)}.\end{aligned} \tag{3.7}$$

This is a one parameter family of maxima, with parameter β. Since we can also act upon these solutions by rotation, we obtain a two parameter family of b's, deforming the obvious critical point $b = 0$, which we can write as

$$b_{A,T}(t) = 2\ln \frac{1 + A\cos(2\pi T)}{1 + A\cos(2\pi(t+T))}, \tag{3.8}$$

where $|A| = |\tanh(2\beta)| < 1$, and T is 1-periodic.

The important point for our purposes is that this family of critical points for $L : \mathcal{D}/\mathrm{Rot} \to \mathbb{R}$ is not a compact set in the W^1 norm for the space of b's. As $A \uparrow 1$ we have the following pointwise limit:

$$2\ln\left(\frac{2}{1 + A\cos(2\pi t)}\right) \to 2\ln\left(\frac{2}{1 + \cos(2\pi t)}\right) = 4\ln(csc(\pi t)). \tag{3.9}$$

This limit is not a continuous function. Therefore, the W^1 norm of this family diverges to $+\infty$ as $A \uparrow 1$.

We thus have, for $q = c\pi^2$,

$$q\frac{\int e^{2b_{A,T}}}{(\int e^{b_{A,T}})^2} - \frac{c}{4}\int b'^2_{A,T} = c\pi^2 \tag{3.10}$$

for all $A < 1$, whereas the energy integral $\int b'^2_{A,T}$ is diverging as $A \uparrow 1$. This implies that for $q > c\pi^2$ the left-hand side will diverge to $+\infty$ as $A \uparrow 1$. This shows that L is not bounded on orbits which are represented by a constant $q > c\pi^2$.

We have now proven (a) for orbits represented by constants and Par_1^-. To complete the proof of (a), we must show that d/dt is not bounded on all other orbits of parabolic and hyperbolic type.

Consider one of our preferred representatives for a hyperbolic orbit as in (5) of Theorem 2.1, which is of the form $c(n\pi)^2 + q_1$. As in the earlier part of this proof, we consider a hyperbolic element $\phi \in \mathrm{PSU}(1,1)$ which is of the form $\phi = R \circ \phi_0 \circ R^{-1}$, where

$$\phi_0 = \begin{pmatrix} c & s \\ s & c \end{pmatrix} \in \mathrm{PSU}(1,1)_1,$$

$c = \cosh(\beta)$, $s = \sinh(\beta)$, and R is a rotation. We choose the rotation so that ϕ has fixed points $\pm e^{2\pi i T}$, $+e^{2\pi i T}$ is an attractive fixed point, and $q_1(e^{2\pi i T}) > 0$. As we calculated above, b_ϕ will equal $b_{A,T}$ as in (3.8). Recalling (3.10) in the case $n = 1$, and remembering the analogous quantity blows up for $n > 1$, we see that (3.2) will be at least

$$c(n\pi)^2 + \frac{\int q_1(\phi(e^{2\pi i t}))\rho(t)^{-4}}{\left(\int \rho(t)^{-2}\right)^2}$$

$$= c(n\pi)^2 + \frac{q_1(e^{2\pi i T})\int \rho^{-4}}{\left(\int \rho^{-2}\right)^2} + \frac{\int \{q_1(\phi) - q_1(e^{2\pi i T})\}\rho^{-4}}{\left(\int \rho^{-2}\right)^2}, \tag{3.11}$$

where $\rho = 1 + A\cos(2\pi(t+T))$. Let $a = A^{-1}$. By explicit calculation

$$\frac{\int \rho^{-4}}{(\int \rho^{-2})^2} = \frac{(3a+2a^3)\pi}{(a-1)^4(1+a)^3\sqrt{(1+a)/(a-1)}} \times \left(\frac{(a-1)^2(a+1)\sqrt{(a+1)/(a-1)}}{2a\pi}\right)^2 \tag{3.12}$$

which is asymptotic to $(5/4\sqrt{2}\pi)(a-1)^{-1/2}$ as $a \downarrow 1$ (i.e. $A \uparrow 1$), and hence diverges. To show that (3.11) diverges, it therefore suffices to show that the second term in (3.11) divided by $(a-1)^{-1/2}$ has zero as a limit as $A \uparrow 1$.

Suppose that we fix $\epsilon > 0$. We have

$$|\phi(z) - \mathrm{e}^{2\pi\mathrm{i}T}| = |\phi_0(\mathrm{e}^{-2\pi\mathrm{i}T}z) - 1| = \frac{|1-\tanh(\beta)||z - \mathrm{e}^{2\pi\mathrm{i}T}|}{|\mathrm{e}^{2\pi\mathrm{i}T} + z\tanh(\beta)|} \tag{3.13}$$

for $z \in S^1$. For $z \neq -\mathrm{e}^{2\pi\mathrm{i}T}$ this tends to 0 as $A \uparrow 1$ (or $\beta \to \infty$). Fix $\delta > 0$ and split the integral in the numerator of the second term of (3.11) into two, one over $|t-T| > \delta$ and the other over $|t-T| < \delta$. For β sufficiently large, and for $|t-T| > \delta$, we will have $|q_1(\phi(\mathrm{e}^{2\pi\mathrm{i}t})) - q_1(\mathrm{e}^{2\pi\mathrm{i}T})| < \epsilon$. Therefore, using (3.12), we see that

$$\left|\frac{\int_{|t-T|>\delta}(q_1(\phi) - q_1(\mathrm{e}^{2\pi\mathrm{i}T}))\rho^{-4}}{(\int \rho^{-2})^2}\right| \le \epsilon\frac{6}{4\sqrt{2}\pi}(a-1)^{-1/2} \tag{3.14}$$

for β sufficiently large. Now consider the other term

$$\frac{\int_{|t-T|<\delta}(q_1(\phi) - q_1(\mathrm{e}^{2\pi\mathrm{i}T}))\rho^{-4}}{(\int \rho^{-2})^2}. \tag{3.15}$$

For t close to T and $A \uparrow 1$, $\rho(t)$ is near 2. Since q is independent of the parameter A and $(\int \rho^{-2})^{-2}$ tends to 0 as $A \uparrow 1$, we can choose δ so that for A near 1, this term is as small as we wish. This concludes the proof that (3.11) diverges as $A \uparrow 1$. Hence $\mathrm{d}/\mathrm{d}t$ is not bounded on hyperbolic orbits.

This same argument works for an orbit of parabolic type Par_n^+, $n > 0$, as well. But in this case there is another more direct argument. In (6) of Theorem 2.1 we displayed a family $\{q_{(n,\alpha)}^+ : 0 < \alpha < 1\}$ which belongs to this orbit. We have

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}t}, c\kappa^* + q_{(n,\alpha)}^+\,\mathrm{d}t^2 \right\rangle = c(n\pi)^2(1+2\alpha)\int_0^1 \frac{1 + \sin(2\pi nt) + 2\alpha\cos^2(2\pi nt)}{(1+\alpha\sin(2\pi nt))^2}\,\mathrm{d}t. \tag{3.16}$$

As $\alpha \uparrow 1$, by considering the integral localized near points where the denominator is near 0, one sees that the values of the integrals tend to $+\infty$. This shows that $\mathrm{d}/\mathrm{d}t$ is not bounded on Par_n^+.

To see that $\mathrm{d}/\mathrm{d}t$ is not bounded on Par_n^-, $n > 1$, we can observe that the representatives $q_{(n,\alpha)}^-$ can be made as close to a constant as we wish, by letting $\alpha \downarrow 0$. Since $\mathrm{d}/\mathrm{d}t$ is not bounded on the orbit of $c(n\pi)^2$, it follows that $\mathrm{d}/\mathrm{d}t$ is not bounded on Par_n^-, $n > 1$. This completes the proof of (a) of Theorem 3.1.

To prove part (b), consider a general Lie group G. Fix an element of the Lie algebra $V \in \mathfrak{g}$. We consider V as a function on a coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$. A point $f \in \mathcal{O}$ is critical for V if and only if the identity in G is critical for the function $L : G \to \mathbb{R} : g \to \langle V, \mathrm{Ad}^*(g)f\rangle$. Since

$$\mathrm{d}L|_1(X) = \langle V, \mathrm{ad}^*(X)f\rangle = \langle [V, X], f\rangle, \tag{3.17}$$

f is critical for V on $\mathcal{O}$ if and only if f vanishes on $\mathrm{ad}(V)(\mathfrak{g})$. Applying this to our context, with $V = \mathrm{d}/\mathrm{d}t$ and $\phi = c\kappa^* + q(\mathrm{d}t)^2$, it is easy to check that q must be constant. □

Remark 3.1.

(a) For the benefit of the reader, we briefly summarize the local analysis of L in (3.3) (see also [17]). We have

$$\mathrm{d}L|_b(B) = 2q\frac{\int \mathrm{e}^{2b}B\int \mathrm{e}^b - \int \mathrm{e}^{2b}\int \mathrm{e}^b B}{(\int \mathrm{e}^b)^3} - \frac{c}{2}\int b'B'. \tag{3.18}$$

Thus if b has two derivatives and is a critical point, then b solves the following integro-differential equation

$$cb'' + 4q\frac{(\int \mathrm{e}^b)\,\mathrm{e}^{2b} - (\int \mathrm{e}^{2b})\,\mathrm{e}^b}{(\int \mathrm{e}^b)^3} = 0. \tag{3.19}$$

Note that this equation is invariant under rotation, i.e. $b(t) \to b(t+T) - b(T)$, as it must be. Also it follows that $b = 0$ is a critical point.

The second derivative is given by

$$\begin{aligned}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^2 L(tb) = 2q\Bigg[\Bigg\{&\int \mathrm{e}^{2b}\dot{b}^2\left(\int \mathrm{e}^b\right)^2 + \int \mathrm{e}^{2b}\dot{b}2\int \mathrm{e}^b\int \mathrm{e}^b\dot{b} - 2\int \mathrm{e}^{2b}\dot{b}\int \mathrm{e}^b\\ &\times\int \mathrm{e}^b\dot{b} - \int \mathrm{e}^{2b}\int \mathrm{e}^b\dot{b}\int \mathrm{e}^b\dot{b} - \int \mathrm{e}^{2b}\int \mathrm{e}^b\int \mathrm{e}^b\dot{b}^2\Bigg\}\\ &\times\left(\int \mathrm{e}^b\right)^4 - \left(\int \mathrm{e}^{2b}\dot{b}\left(\int \mathrm{e}^b\right)^2 - \int \mathrm{e}^{2b}\int \mathrm{e}^b\int \mathrm{e}^b\dot{b}\right)\\ &\times 4\left(\int \mathrm{e}^b\right)^3\int \mathrm{e}^b\dot{b}\Bigg]\left(\int \mathrm{e}^b\right)^{-8} - \frac{c}{2}\int \dot{b}'^2.\end{aligned} \tag{3.20}$$

At a general point this is pretty useless. However, at the critical point $b = 0$, we have

$$\mathrm{Hess}|_{b=0}(\dot{b}) = 2q\int \dot{b}^2 - \frac{c}{2}\int \dot{b}'^2 = \sum_{n=1}^{\infty}\left\{2q - \frac{c}{2}(2\pi n)^2\right\}(\alpha_n^2 + \beta_n^2), \tag{3.21}$$

where $\dot{b} = \sum\{\alpha_n \sin(2\pi nt) + \beta_n\{\cos(2\pi nt) - 1\}\}$.

This Hessian, at the point $b = 0$, is negative semidefinite if and only if $q \leq c\pi^2$. More generally at each bifurcation point, as ρ increases, we pick up another positive eigenvalue for the Hessian, always with multiplicity 2.

(b) Theorem 3.1 leads to an intuitive picture for the orbits from a Morse-theoretic point of view. The orbits below and including the universal Teichmuller orbit are infinite dimensional analogues of $t^2 - |x|^2 = 1$, with $t < 0$. The orbit Par_1^- is an infinite dimensional analogue of $t^2 - |x|^2 = 0$, $t < 0$ (since the maximum is missing). The hyperbolic orbits are analogous to the one sheeted hyperboloid $t^2 - |x|^2 = -1$. The novel feature is that the elliptic orbits Ell_n above the universal Teichmuller orbit are saddle-shaped, with $2(n-1)$ directions pointing up.

4. Virasoro structure and Kirillov's orbit correspondence

4.1. Virasoro structure

The complex Virasoro algebra *Vir* is given in terms of generators and relations by

$$\mathrm{Vir} = \left(\sum_{n\in\mathbb{Z}} \mathbb{C}L_n\right) \oplus \mathbb{C}\mathbf{K}, \tag{4.1}$$

where

$$[L_n, L_m] = (n-m)L_{n+m} + \tfrac{1}{12}n(n^2-1)\delta(n+m)\mathbf{K}, \quad [L_n, \mathbf{K}] = 0. \tag{4.2}$$

Given our choice of cocycle (1.8), the relation between Vir and $\hat{\mathcal{V}}^{\mathbb{C}}$ is given by the mapping $\mathrm{Vir} \to \hat{\mathcal{V}}^{\mathbb{C}}$

$$\mathbf{K} \to 12\pi\mathrm{i}\kappa, \qquad L_n \to \frac{1}{2\pi\mathrm{i}} \mathrm{e}^{2\pi\mathrm{i}nt}\frac{\mathrm{d}}{\mathrm{d}t}, \quad n \neq 0, \qquad L_0 \to \frac{1}{2\pi\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}t} + \frac{1}{24}\mathbf{K}. \tag{4.3}$$

(There is freedom in the choice of cocycle defining $\hat{\mathcal{V}}$, and we could eliminate the 12π and the shift in L_0 by replacing (1.8) by

$$\frac{\zeta(-1)}{2\pi\mathrm{i}} \int \nabla(\xi)\,\mathrm{d}\nabla(\eta), \tag{4.4}$$

where $\nabla(f(\partial/\partial z)) = (\partial f/\partial z)(\partial/\partial z))$.

The Virasoro algebra has a triangular decomposition, in the technical sense of [9], where

$$\mathfrak{n}^{\pm} = \sum_{\pm n>0} \mathbb{C}L_n \quad \text{and} \quad \hat{\mathfrak{h}} = \mathbb{C}L_0 \oplus \mathbb{C}\mathbf{K}. \tag{4.5}$$

The roots are $\{\alpha_n = n\alpha_1 : n \in \mathbb{Z} \setminus \{0\}\}$, where

$$\alpha_1(L_0) = -1, \quad \alpha_1(\mathbf{K}) = 0, \tag{4.6}$$

and L_n spans the root space for α_n. It is natural to define

$$\delta = \frac{1}{2}\left(\sum_{n>0} n\alpha_1\right) = \frac{1}{2}\zeta(-1)\alpha_1 = -\frac{1}{24}\alpha_1. \tag{4.7}$$

For each $n > 0$, there is an embedding

$$\mathrm{d}i_n : sl(2,\mathbb{C}) \to \mathrm{vir} : \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \to f_n = \frac{1}{n} L_{-n},$$
$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \to h_n = \frac{2}{n} L_0 + \frac{1}{12n}(n^2-1)\mathbf{K}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \to e_n = \frac{1}{n} L_n. \tag{4.8}$$

Geometrically this corresponds to the following. The group of projective transformations of $\hat{\mathbb{C}}$ which map the circle to itself is the subgroup $\mathrm{PSU}(1,1) \subset \mathrm{PSL}(2,\mathbb{C})$, where

$$\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \cdot z' = \frac{az' + b}{\bar{b}z' + \bar{a}}. \tag{4.9}$$

For $n \geq 1$ there is an n-fold covering map

$$S^1 \to S^1 : z \to z' = z^n \tag{4.10}$$

and the diffeomorphisms of z which cover the projective transformations of z' form a group $\mathrm{PSU}(1,1)^{(n)}$ which is an n-fold covering

$$0 \to \mathbb{Z}_n \to \mathrm{PSU}(1,1)^{(n)} \to \mathrm{PSU}(1,1) \to 0. \tag{4.11}$$

The map di_n, modulo the center, is the complexification of the differential of the embedding

$$i_n : \mathrm{PSU}(1,1)^{(n)} \to \mathcal{D}. \tag{4.12}$$

It is tempting to think of h_n as the coroot corresponding to the root α_n, $\mathrm{d}i_n$ as the embedding corresponding to this coroot, and so on, analogous to the Kac–Moody case. However, there are fundamental differences. Vir does not have an Ad-invariant form (so that Vir is not a generalized Kac–Moody algebra, in the sense of Borcherds); the category of highest weight representations is not semisimple and parabolic subalgebras are simple, rather than of the form semisimple $\propto$ nilradical. Except in the cases $n = 1, 2$, the embeddings $\mathrm{d}i_n$ cannot be globalized: $\mathrm{PSU}(1,1)^{(n)}$ does not have a complexification for $n > 2$, and this implies that $\mathcal{D}$ does not have a complexification.

4.2. *Naive orbit correspondence*

In our context we know the unitary highest weight representations of Vir, and we want to set up an orbit correspondence. The Fourier transform of an orbit requires regularization, as we will see in Section 5 below, so we first consider the moment map method of setting up a correspondence with coadjoint orbits.

Suppose that G acts irreducibly and unitarily on a Hilbert space $\mathcal{H}$. If the corresponding Lie algebra representation $\mathrm{d}\pi$ is of highest weight type, with highest weight vector v_0, then the moment map method of obtaining a coadjoint orbit proceeds as follows. Consider the orbit of lowest weight vectors

$$G \cdot \mathbb{P}(\bar{v}_0) \subset \mathbb{P}(\bar{\mathcal{H}}) \tag{4.13}$$

with the Hamiltonian G-structure induced by the Fubini-Study metric; via the moment map

$$G \cdot \mathbb{P}(\bar{v}_0) \to \mathfrak{g}^* : \mathbb{P}(\bar{v}) \to \frac{\mathrm{i}}{2\pi} \frac{\langle \mathrm{d}\pi(\cdot) \cdot v, v \rangle}{\langle v, v \rangle}, \tag{4.14}$$

this orbit is identified with a unique coadjoint orbit $\mathcal{O}$. Note that in this correspondence, given a Lie algebra element, its classical energy (i.e. its value at a point of the coadjoint orbit) is the same as its quantum mechanical energy in the state defined by the corresponding highest weight vector.

In good cases (e.g. if G is compact), if we identify $\mathcal{O}$ and $G \cdot \mathbb{P}(\bar{v}_0)$ and let $\mathcal{T}$ denote the tautological bundle of $\mathbb{P}(\bar{\mathcal{H}})$ restricted to $\mathcal{O}$, then $\mathcal{T}^*$ is the line bundle corresponding to the Kirillov symplectic structure, and there is an equivariant isomorphism of Borel–Weil

$$\mathcal{H} \to H^0(\mathcal{O}, \mathcal{T}^*) : w \to s_w, \tag{4.15}$$

where s_w is the section of $\mathcal{T}^*$ given by $s_w(\bar{v}) = \langle w, \bar{v} \rangle$.

Let $H(\mathbf{c}, \mathbf{h})$ denote the unique irreducible highest weight module of Vir corresponding to the character of $\mathfrak{h}$ determined by $\mathbf{K} \cdot v_0 = \mathbf{c} v_0$ and $L_0 \cdot v_0 = \mathbf{h} v_0$. The conditions under which this representation is unitarizable are well-known: either $\mathbf{c} \geq 1$ and $\mathbf{h} \geq 0$, or $\mathbf{c} = 1 - (6/m(m+1))$, $m = 2, 3, \ldots$, and $\mathbf{h} = (((m+1)p - mq)^2 - 1)/4m(m+1)$, for some $0 < q \leq p < m$.

Proposition 4.1. *Suppose that $H(\mathbf{c}, \mathbf{h})$ is unitarizable. Then there is a $\mathcal{D}$-equivariant isomorphism of Hamiltonian $\mathcal{D}$-spaces*

$$\mathcal{O}(c\kappa^* + q) \leftrightarrow \mathcal{D} \cdot \mathbb{P}(\bar{v}_0) \subset \mathbb{P}(\bar{L}),$$

where $c = \mathbf{c}/24\pi^2$ and $q = c\pi^2(1 - (24\mathbf{h}/\mathbf{c}))$. In particular, for $\mathbf{h} = 0$ we obtain the universal Teichmuller orbit, and for $\mathbf{h} > 0$ we obtain an orbit below the Teichmuller orbit, in the sense of Fig. 2, where $\mathrm{d}/\mathrm{d}t$ is bounded.

This follows directly from the identity

$$\langle c\kappa^* + q, x \rangle = \frac{\mathrm{i}}{2\pi} \frac{\langle \mathrm{d}\pi(x) v_0, v_0 \rangle}{\langle v_0, v_0 \rangle}, \quad x \in \hat{\mathcal{V}} \tag{4.16}$$

taking $x = \kappa$ yields the value for c; taking $x = L_n$, $n \neq 0$, shows that q is constant and taking $x = L_0$ yields the constant value for q.

5. Orbital integrals

In Section 5.3 we will heuristically calculate the Fourier transform of a $\mathcal{D}$ orbit $\mathcal{O}$ in $c\kappa^* + Q$, where $c > 0$. In the calculation we will formally manipulate a fictional Haar measure for $\mathcal{D}$. The point of Section 5.1 is to discuss what we might mean by such an object. The ideal possibility is that such a object exists as a limit of well-defined quasi-invariant measures which are asymptotically invariant in some sense. It turns out that there does exist a left-invariant Haar measure of this sort (a limit of the Malliavin–Shavgulidze measures considered in [11]). However, to obtain a mathematically well-defined transform of $\mathcal{O}$

it seems necessary to consider another possibility for Haar measure, which is somewhat surprising.

In Section 5.4 we will rigorously discuss the integrals which we arrive at in Section 5.3. When we calculate their values, there is another surprise. Instead of encountering something reminiscent of characters, we find something related to the $\hat{A}$ and L genuses, which is related to $N = 1$ supersymmetry, and which we used in Section 2 to explain the form of the coadjoint action. This may or may not be a coincidence.

5.1. *"Haar measures" for $\mathcal{D}$*

The group $\tilde{\mathcal{D}}$ is the product of the two subgroups $\tilde{\mathcal{D}}_0$ (the stability subgroup of $0 \in \mathbb{R}$, where we view $\tilde{\mathcal{D}}$ acting on $\mathbb{R}$ as in (1.5)) and $\mathbb{R}$, the translations. These two subgroups do not commute. We therefore consider two coordinate systems

$$\mathbb{R} \times \mathrm{Path}^{0,0}\mathbb{R} \leftrightarrow \tilde{\mathcal{D}} \leftrightarrow \mathrm{Path}^{0,0}\mathbb{R} \times \mathbb{R}, \tag{5.1}$$

$$(t, B) \leftrightarrow \Psi_{t,B} = \Psi = \Psi_{b,\tau} \leftrightarrow (b, \tau), \tag{5.2}$$

where

$$t + \frac{\int_0^x \mathrm{e}^B}{\int \mathrm{e}^B} = \Psi(x) = \frac{\int_0^{\tau+x} \mathrm{e}^b}{\int \mathrm{e}^b} \tag{5.3}$$

and we initially think of $\mathrm{Path}^{0,0}\mathbb{R}$ as consisting of *smooth* 1-periodic functions $B : \mathbb{R} \to \mathbb{R}$ with $B(0) = 0$ (we will necessarily have to relax this smoothness condition when we consider Wiener measure below). The left coordinates are given by $t = \Psi(0)$, $B = \ln \Psi' - \ln \Psi'(0)$; the formulas for τ and b are not quite as explicit. In the left coordinate multiplication is given by the straightforward formula

$$(t, B)(s, C) = \left(t + \frac{\int_0^s \mathrm{e}^B}{\int \mathrm{e}^B}, B\left(s + \frac{\int_0^{(\cdot)} \mathrm{e}^C}{\int \mathrm{e}^C}\right) + C(\cdot) - B(s)\right). \tag{5.4}$$

The multiplication in the right system is given by

$$(b, \tau)(c, \eta) = (\mathcal{B}, \mathcal{T}), \tag{5.5}$$

where

$$\mathcal{B}(x) = b\left(\tau + \frac{\int^{\eta-\tau+x} \mathrm{e}^c}{\int \mathrm{e}^c}\right) + c(\eta - \tau + x) - b\left(\tau + \frac{\int^{\eta-\tau} \mathrm{e}^c}{\int \mathrm{e}^c}\right) - c(\eta - \tau) \tag{5.6}$$

and $\mathcal{T}$ is determined implicitly by the relation

$$\tau = -\frac{\int^{\eta-\mathcal{T}} \mathrm{e}^c}{\int \mathrm{e}^c}. \tag{5.7}$$

There are corresponding left and right coordinate systems for $\mathcal{D}$, where $\mathbb{R}$ is replaced by $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

Let $\nu_T^{0,0}$ denote the pinned Brownian measure on the path space $\mathrm{Path}_{C^0}^{0,0}\mathbb{R}$ with temperature or variance T. This is the Gaussian measure which corresponds to the Cameron–Martin Hilbert space consisting of W^1 paths with norm $(1/T)\int b'^2$, where $\int f$ means $\int_0^1 f$, unless otherwise specified.

Lemma 5.1. *In terms of the left and right coordinate systems above, for either $\tilde{\mathcal{D}}$ or $\mathcal{D}$, we have*

$$\mathrm{d}t \times \mathrm{d}\nu_T^{0,0}(B) = \frac{\mathrm{e}^{b(\tau)}}{\int \mathrm{e}^b}\, \mathrm{d}\nu_T^{0,0}(b) \times \mathrm{d}\tau, \tag{5.8}$$

$$\left(\int \mathrm{e}^B\right) \mathrm{d}t \times \mathrm{d}\nu_T^{0,0}(B) = \mathrm{d}\nu_T^{0,0}(b) \times \mathrm{d}\tau. \tag{5.9}$$

Proof. This follows directly from the relations

$$t = \frac{\int^\tau \mathrm{e}^b}{\int \mathrm{e}^b}, \qquad B(x) = b(\tau + x) - b(\tau). \qquad \square \tag{5.10}$$

The probability measure $\mathrm{d}\nu_T^{0,0}(b) \times \mathrm{d}\tau$ in $\mathcal{D}_{C^1}$, denoted $\mathrm{d}\nu_T$, is what we referred to as the Malliavin–Shavgulidze measure at temperature T in Chapter 4 of [11]. (Note: in [11] we used inverse temperature β as the parameter, and we used $\theta = 2\pi t$ as our preferred coordinate for S^1, so there are some nuances in translating between our present notation and that in [11].) This measure is well-known to be left quasi-invariant with respect to $\tilde{\mathcal{D}}$ (due originally to the Malliavins and Shavgulidze, in a much more general context), and the transformation properties are what one would expect based upon purely formal calculations (see (3.2.12) of Chapter 4 of [11], or Proposition 5.1 below).

Let $\mathrm{d}V$ denote a formal Lebesgue measure for the linear space $\mathrm{Path}^{0,0}\mathbb{R}$. Since this is an infinite dimensional space, this notion is ambiguous, even at the formal level. What is needed is a Cameron–Martin space which gives us an idea of what we mean by volume in the infinite dimensional limit. In the present context we take the Cameron–Martin space mentioned previously, consisting of W^1 paths, with norm given by the energy $\int b'^2$ (note this is not natural geometrically—it depends upon the choice of coordinate).

We claim that we can interpret $\mathrm{d}V(b) \times \mathrm{d}\tau$ as a left Haar measure. To verify this in a formal way, we first calculate (using the left coordinate system, where multiplication is more explicit)

$$\begin{aligned} L_{(t,B)}^*(\mathrm{d}s \times \mathrm{d}V(C)) &= \frac{\mathrm{e}^{B(s)}}{\int \mathrm{e}^B}\, \mathrm{d}s \times \mathrm{Jacobian}[C \to C + B(\Psi_{s,C}) - B(s)]\, \mathrm{d}V(C) \\ &= \frac{\mathrm{e}^{B(s)}}{\int \mathrm{e}^B}\, \mathrm{d}s \times \mathrm{d}V(C). \end{aligned} \tag{5.11}$$

The fact that the Jacobian is 1 (in a precise mathematical sense, using the Cameron–Martin space) is verified on pages 110–111 of [11]. In support of this note that

$$\rho((t, B); (s, C)) = \frac{\mathrm{e}^{B(s)}}{\int \mathrm{e}^B} = \Psi_{t,B}'(\Psi_{s,C}(0)) \tag{5.12}$$

satisfies the requisite cocycle identity $\rho(\psi \circ \phi, \eta) = \rho(\psi; \phi \circ \eta)\rho(\phi; \eta)$, as a direct consequence of the chain rule.

We now calculate

$$L^*_{(s,C)}(\mathrm{d}V(b) \times \mathrm{d}\tau) = L^*_{(s,C)}\left(\left(\int \mathrm{e}^B\right) \mathrm{d}t \times \mathrm{d}V(B)\right)$$
$$= \frac{\int \mathrm{e}^{B+b(\Psi_{t,B})}}{\mathrm{e}^{b(t)}} \frac{\mathrm{e}^{b(t)}}{\int \mathrm{e}^b} \mathrm{d}t \times \mathrm{d}V(B) \tag{5.13}$$

(where we have used (5.11)). Now

$$\int \mathrm{e}^{B+b(\Psi_{t,B})} = \int \mathrm{e}^{B(\Psi^{-1}_{t,B})+b} \frac{1}{\Psi'_{t,B}(\Psi^{-1}_{t,B})} = \int \mathrm{e}^B \int \mathrm{e}^b, \tag{5.14}$$

because

$$\frac{\mathrm{e}^{B(\Psi^{-1}_{t,B})}}{\Psi'_{t,B}(\Psi^{-1}_{t,B})} = \frac{\mathrm{e}^{B(\Psi^{-1}_{t,B})}}{\left(\mathrm{e}^{B(\Psi^{-1}_{t,B})} / \int \mathrm{e}^B\right)} = \int e^B. \tag{5.15}$$

Plugging this into (5.13) shows that $\mathrm{d}V(b) \times \mathrm{d}\tau$ is formally a left Haar measure.

In Section 5.4 we will show that $\mathrm{d}\nu_T(b) \times \mathrm{d}\tau$ is asymptotically invariant, as $T \uparrow \infty$, which is a more precise sense in which $DV(b) \times \mathrm{d}\tau$ can be viewed as a (weak) Haar measure.

We now want to formally consider right invariance properties of the measures above. We formally calculate

$$R^*_{(s,C)}(\mathrm{d}t \times \mathrm{d}V(B)) = \mathrm{d}t \times \det(B \to C + B(\Psi_{s,C}) - B(s))\,\mathrm{d}V(B). \tag{5.16}$$

The affine action on B is the coordinate expression for the right action of $(s, C) \in \tilde{\mathcal{D}}$ on the right coset space $\mathbb{R} \setminus \tilde{\mathcal{D}}$, hence the linear part of the action is a representation. Thus one can argue that the determinant should define a character of $\mathcal{D}$, hence since $\mathcal{D}$ is simple the determinant must be trivial. Assuming the soundness of this argument, we conclude that

$$\mathrm{d}m(\psi) = \mathrm{d}t \times \mathrm{d}V(B) \tag{5.17}$$

is a right Haar measure. This argument is questionable because the determinant simply does not exist from the Cameron–Martin point of view.

We have drawn the following conclusions, using purely formal reasoning: (1) the expression $\mathrm{d}V(b) \times \mathrm{d}\tau$ is formally a left Haar measure; (2) the expression $\mathrm{d}t \times \mathrm{d}V(B)$ might be a right Haar measure. For (1) and (2) to hold simultaneously, we must give up the formal idea that (either right or left) Haar measure is unique for $\mathcal{D}$. Thus in particular either $\mathrm{d}t \times \mathrm{d}V(B)$ or the inversion of $\mathrm{d}V(b) \times \mathrm{d}\tau = \left(\int \mathrm{e}^B\right) \mathrm{d}t \times \mathrm{d}V(B)$ can be taken as right Haar measure.

Given that uniqueness of Haar measure fails, there is an ambiguity about what we might mean by the modular function. This is important in orbit theory for finite dimensional nonunimodular groups (see p. 450 of [7]), where the modular function is related to the j function in (1.3).

In the case of $\hat{\mathcal{D}}$, we can formally compute the j function using a ζ-function regularization. Recall from Section 4 that the complex roots for the adjoint action of $\mathfrak{h}$ on $\hat{\mathcal{V}}$ are integral multiples of a single root α_1 with $\alpha_1(L_0) = -1$. Therefore (1.3) equals

$$\begin{aligned} j(yL_0) &= \prod_{n>0} \frac{e^{ny/2} - e^{-ny/2}}{ny} = \prod_{n>0} (ny)^{-1}(1-q^n)q^{-n/2} \\ &= e^{-\sum \log(n)} y^{-\sum 1} q^{-(1/2)\sum n} \prod (1-q^n) = e^{\zeta'(0)} y^{-\zeta(0)} q^{-(1/2)\zeta(-1)} \phi(q) \\ &= \frac{1}{\sqrt{2\pi}} y^{1/2} \eta(q), \end{aligned} \tag{5.18}$$

where $q = e^{-y}$.

5.2. Heuristic calculation of Fourier transformations, using left Haar measure

In this section we will work with the formal expression $dm_L(\psi) = dt \times dV(B)$, for which we have strong reason to interpret as a formal left Haar measure on $\mathcal{D}$, by (5.1).

Our goal is to formally compute the Laplace transform of the orbit $\mathcal{O}$, which we naively expect to define an Ad-invariant function on $\hat{\mathcal{V}}$. Initially, we suppose that $y(d/dt) \in \mathcal{V}$ is arbitrary, but eventually we will focus on the case in which y is constant. We formally calculate

$$\begin{aligned} \int_{\mathcal{O}} e^{\langle y(d/dt) + is\kappa, \cdot \rangle} \frac{1}{\infty!} \omega^\infty &= \mathrm{Vol}(\mathcal{O})\, e^{isc} \int_{\mathcal{D}} e^{\langle y(d/dt), \psi \cdot (q(dt)^2 + c\kappa^*)\rangle} \, dm_L(\psi) \\ &= \mathrm{Vol}(\mathcal{O})\, e^{isc} \int e^{\int y\{q(\psi^{-1})(\Psi^{-1})'^2 + (c/2)S(\Psi^{-1})\}\, dt} \, dm_L(\psi). \end{aligned} \tag{5.19}$$

Now $S(\Psi^{-1}) = -S(\Psi) \circ \Psi^{-1}(\Psi^{-1})'^2$ and $S(\Psi) = b'' - \frac{1}{2}b'^2$. Thus

$$\begin{aligned} \int yS(\Psi^{-1})\, dt &= -\int y(\Psi) \left\{ b'' - \frac{1}{2}b'^2 \right\} \frac{1}{\Psi'^2}\, d\Psi = -\int \frac{y(\Psi)}{\Psi'} \left\{ b'' - \frac{1}{2}b'^2 \right\} d\tau \\ &= \int \left\{ \left(\frac{y'(\Psi)\Psi'^2 - y(\Psi)\Psi''}{\Psi'^2} \right) b' + \frac{1}{2} \frac{y(\Psi)}{\Psi'} b'^2 \right\} d\tau \\ &= -\int y''(\Psi)\Psi' b - \frac{1}{2} \int \frac{y(\Psi)}{\Psi'} b'^2 \, d\tau, \end{aligned} \tag{5.20}$$

where in arriving at the last line we used the identity $b'\Psi' = \Psi''$.

Thus (5.19) equals

$$\mathrm{Vol}(\mathcal{O})\, e^{isc} \int e^{\int \{(y(\Psi)/\Psi')q - (c/2)y''(\Psi)\Psi' b\}} e^{-(c/4)\int (y(\Psi)/\Psi')b'^2} \, dV(b)\, d\tau. \tag{5.21}$$

Now consider the case in which y is a positive constant. We need to make sense of the expression

$$\frac{1}{\mathcal{Z}} e^{-(1/2)\int (b'^2/\Psi')} \, dV(b)\, d\tau \tag{5.22}$$

as a measure. It appears that this simply cannot be done, because of the nonlocal nature of $\Psi' = \mathrm{e}^b/(\mathrm{e}^b)$. If one attempts to formally compute the Radon–Nikodym derivative for left translation, one arrives at complete nonsense.

We conclude that while $\mathrm{d}m_{\mathrm{L}} = \mathrm{d}V(b) \times \mathrm{d}\tau$ is a reasonable candidate for a left Haar measure, it simply is not the right one to use for orbit theory.

5.3. Heuristic calculation of Fourier transformations, using right Haar measure

In this section we will work with the formal expression $\mathrm{d}m_{\mathrm{R}}(\psi) = \mathrm{d}t \times \mathrm{d}V(B)$, for which we have some vague reason to interpret as a formal right Haar measure on $\mathcal{D}$.

We again initially suppose that $y(\mathrm{d}/\mathrm{d}t) \in \mathcal{V}$ is arbitrary. We formally calculate:

$$\begin{aligned}
&\int_{\mathcal{O}} \mathrm{e}^{\langle y(\mathrm{d}/\mathrm{d}t)+\mathrm{i}s\kappa,\cdot\rangle} \frac{1}{\infty!}\omega^{\infty} \\
&= \mathrm{Vol}(\mathcal{O})\,\mathrm{e}^{\mathrm{i}sc} \int_{\mathcal{D}} \mathrm{e}^{\langle y(\mathrm{d}/\mathrm{d}t),\psi^{-1}\cdot(q(\mathrm{d}t)^2+c\kappa^*)\rangle}\,\mathrm{d}m_{\mathrm{R}}(\psi) \\
&= \mathrm{Vol}(\mathcal{O})\,\mathrm{e}^{\mathrm{i}sc} \int \mathrm{e}^{\int y\{q(\psi)\Psi'^2+(c/2)(B''(t)-(1/2)B'^2(t))\}\,\mathrm{d}t}\,\mathrm{d}m_{\mathrm{R}}(\psi) \\
&= \mathrm{Vol}(\mathcal{O})\,\mathrm{e}^{\mathrm{i}sc} \int \mathrm{e}^{\int((c/2)y''B+yq(\psi)(\mathrm{e}^{2B}/(\int \mathrm{e}^B)^2))}\,\mathrm{e}^{-(c/4)\int yB'^2}\,\mathrm{d}m_{\mathrm{R}}(\psi).
\end{aligned} \tag{5.23}$$

Remark 5.1.

(1) In the first equality we put ψ^{-1}, instead of ψ, so that to this point we have only used the assumption that $\mathrm{d}m_{\mathrm{R}}$ is a right Haar measure.

(2) Switching the integral over $\mathcal{O}$ to an integral over $\mathcal{D}$ is permissible for orbits of types 1–3 in Theorem 2.1, but questionable for the others, because in these latter cases the stability subgroups are noncompact. The factor $\mathrm{Vol}(\mathcal{O})$ is the ratio of $(1/\infty!)\omega^{\infty}$ and $\mathrm{d}V(B)$, which should be a constant depending only upon the orbit, by invariance of the two volume forms.

(3) Note that we simply deleted the B' term appearing in (5.23), invoking the fact that B corresponds to a diffeomorphism of S^1, hence should have a periodic derivative. This is essential for the next step in our argument, which is to write $B(t) = w(t) - tw(t)$, where $w(t)$ is a Brownian motion, since w is not differentiable. This may explain why we do not obtain characters at the end of this paper. Unfortunately, we simply do not see how to retain this term and obtain mathematically well-defined integrals.

We now specialize to the case in which y is a positive constant. We then claim that (5.23) equals

$$\begin{aligned}
&\mathrm{Vol}(\mathcal{O})\,\mathrm{e}^{\mathrm{i}sc} \int \mathrm{e}^{y(\int q(\psi)\,\mathrm{e}^{2B}/(\int \mathrm{e}^B)^2)}\,\mathrm{e}^{-(yc/4)\int B'^2}\,\mathrm{d}V(B)\,\mathrm{d}t \\
&= \mathrm{Vol}(\mathcal{O})\,\mathrm{e}^{\mathrm{i}sc}\frac{1}{\mathcal{Z}(yc)} \iint \exp\left(y\frac{\int q(\psi)\,\mathrm{e}^{2B}}{(\int \mathrm{e}^B)^2}\right)\mathrm{d}\nu^{0,0}_{(yc/2)^{-1}}(B)\,\mathrm{d}t.
\end{aligned} \tag{5.24}$$

where $\nu_T^{0,0}$ denotes the conditioned Wiener measure as in Section 5.1. Formally we have

$$\mathrm{d}\nu_T^{0,0}(B) = (2\pi T)^{-\infty/2}\,\mathrm{e}^{-(1/2T)\int B'^2}\,\mathrm{d}V(B). \tag{5.25}$$

If we interpret the ∞ to mean $1+1+1+\cdots$, we could interpret this as $\zeta(0)$. Thus we could interpret $\mathcal{Z}(yc) = C(yc)^{-\zeta(0)}$, where C is independent of all the other parameters. Accordingly, assuming $y>0$, we will rewrite (5.24) as

$$\mathrm{Vol}(\mathcal{O})\,\mathrm{e}^{\mathrm{i}sc}(yc)^{\zeta(0)}\iint \exp\left(y\frac{\int q(\psi)\,\mathrm{e}^{2B}}{(\int \mathrm{e}^{B})^2}\right)\mathrm{d}\nu_T^{0,0}(B)\,\mathrm{d}t. \tag{5.26}$$

where $T=2/yc$. In the next section we will see that this integral does exist, for orbits for which $\mathrm{d}/\mathrm{d}t$ is bounded and the stability subgroup is compact.

Thus, in contrast to what developed in Section 5.2, while our argument that $\mathrm{d}t\times \mathrm{d}V(B)$ is a right Haar measure was not so strong, it does at least have the virtue of leading to integrals that can be analyzed.

5.4. *Rigorous discussion of integrals*

As in [3] our basic method of establishing the existence and computing the integrals above is to use a change of variables formula for Wiener measure.

Suppose that $f\in \mathrm{Path}^{0,0}_{C^2}\mathbb{R}$. Consider the transformation of the space $\mathrm{Path}^{0,0}\mathbb{R}$ given by

$$B(t)\to B(t)+f(\Psi_{0,B}(t)), \tag{5.27}$$

where $(0,B)$ and $\Psi_{0,B}$ correspond as in (5.1)–(5.3).

Proposition 5.1. *For f as above*

$$\frac{\mathrm{d}\nu_T^{0,0}(B+f(\Psi_{0,B}))}{\mathrm{d}\nu_T^{0,0}(B)} = \exp\left(-\frac{1}{2T}\left\{\int (f'(\Psi_{0,B})^2 - 2f''(\Psi_{0,B}))\left(\frac{\mathrm{e}^B}{\int \mathrm{e}^B}\right)^2 + 2f'|_0^1\right\}\right),$$

implying

$$\int \exp\left(-\frac{1}{T}\left\{\int \frac{1}{2}(f'(\psi_{0,B})^2 - f''(\psi_{0,B}))\left(\frac{\mathrm{e}^B}{\int \mathrm{e}^B}\right)^2\right\}\right)\mathrm{d}\nu_T^{0,0}(B) = \exp(T^{-1}f'|_0^1).$$

The Radon–Nikodym formula is a consequence of a nontrivial but standard formula for transformations of Wiener space due to Gross (see pp. 110–111 of [11], with f in place of b_ϕ, and just remembering that our f no longer satisfies the special condition $f'|_0^1=0$). But using a formal calculation it is easy to explain why the result is true. Formally the Radon–Nikodym derivative is given by (with $\psi=\psi_{0,B}$)

$$\frac{\mathrm{d}\nu_T^{0,0}(B+f(\psi))}{\mathrm{d}\nu_T^{0,0}(B)} = \frac{\exp(-(1/2T)\int (B+f(\psi))'^2)}{\exp(-(1/2T)\int B'^2)}, \tag{5.28}$$

$$\frac{\mathrm{d}\nu_T^{0,0}(B+f(\psi))}{\mathrm{d}\nu_T^{0,0}(B)} = \exp\left(-\frac{1}{2T}\int (2B'f(\psi)' + f(\psi)'^2)\right), \tag{5.29}$$

Note that $\psi' = \mathrm{e}^B / \int \mathrm{e}^B$. Now calculate

$$\int (2B'f(\psi)' + f(\psi)'^2) = \int \left\{\left(f'(\psi)\frac{\mathrm{e}^B}{\int \mathrm{e}^B}\right)^2 + 2f'(\psi)\left(\frac{\mathrm{e}^B}{\int \mathrm{e}^B}\right)'\right\}, \tag{5.30}$$

$$\int (2B'f(\psi)'+f(\psi)'^2) = \int \left\{\left(f'(\psi)\frac{\mathrm{e}^B}{\int \mathrm{e}^B}\right)^2 - 2f''(\psi)\left(\frac{\mathrm{e}^B}{\int \mathrm{e}^B}\right)^2\right\} + 2f'(\psi)|_0^1. \tag{5.31}$$

which formally implies the Radon–Nikodym formula. The second part of Proposition 5.1 follows from the first by integrating the Radon–Nikodym formula.

We can use this transformation formula to obtain an algorithm for computing the integrals of the preceding subsection, especially Remark 5.1. For each fixed t, solve the differential equation

$$h''(t,x) - \frac{1}{2}h'^2 2(t,x) = \frac{2}{c}q(x+t) \tag{5.32}$$

subject to the boundary condition $h(t,0) = h(t,1) = 0$. After the substitution $H = \mathrm{e}^{-h/2}$, this equation is equivalent to the Hill's equation $cH''(x) + q(t+x)H(x) = 0$, subject to the boundary condition $H(0) = H(1) = 1$, where of course H must be positive.

We then calculate (where, as in Section 5.3, $\psi = \psi_{t,B}$)

$$\begin{aligned}
&\iint \exp\left(y\frac{\int q(\psi)\,\mathrm{e}^{2B}}{(\int \mathrm{e}^B)^2}\right) \mathrm{d}\nu^{0,0}_{(yc/2)^{-1}}(B)\,\mathrm{d}t \\
&= \iint \exp\left(\frac{yc}{2}\frac{\int (h''(t,\psi) - (1/2)h'^2(t,\psi))\,\mathrm{e}^{2B}}{(\int \mathrm{e}^B)^2}\right) \mathrm{d}\nu^{0,0}_{(yc/2)^{-1}}(B)\,\mathrm{d}t \\
&= \int \mathrm{e}^{(yc/2)(h'(t,1)-h'(t,0))}\,\mathrm{d}t.
\end{aligned} \tag{5.33}$$

Now consider the special case in which q is constant. In this case the t-translation is irrelevant. We have

$$H = \cos\left(\sqrt{\frac{q}{c}}x\right) + \frac{1-\cos(\sqrt{q/c})}{\sin(\sqrt{q/c})}\sin\left(\sqrt{\frac{q}{c}}x\right), \tag{5.34}$$

(which is positive provided $q < c\pi^2$), and

$$h'(0) = -h'(1) = -2H'(0) = \sqrt{\frac{q}{c}}\frac{1-\cos(\sqrt{q/c})}{\sin(\sqrt{q/c})}. \tag{5.35}$$

Thus for $q < c\pi^2$ we can apply (Proposition 5.1) to obtain the following.

Corollary 5.1. *Suppose* $c, y > 0$, *and* q *is constant. Then*

$$\iint \exp\left(yq\frac{\int \mathrm{e}^{2B}}{(\int \mathrm{e}^{B})^2}\right) \mathrm{d}\nu^{0,0}_{(yc/2)^{-1}}(B)\,\mathrm{d}t = \exp\left(yc\sqrt{\frac{q}{c}}\frac{1-\cos(\sqrt{q/c})}{\sin(\sqrt{q/c})}\right),$$

provided $q < c\pi^2$, *and diverges otherwise.*

Note that the integrand on the left-hand side of Corollary 5.1 is an increasing function of q, so that as soon as the integral diverges at $q = c\pi^2$, it diverges for all $q > c\pi^2$.

Remark 5.2.

(1) When $q = c\pi^2$ the integral presumably diverges because we really should have an integral over $\mathrm{PSU}(1,1) \setminus \mathcal{D}$, rather than $\mathrm{Rot} \setminus \mathcal{D}$. For $q > c\pi^2$ the divergence is at least heuristically linked to the unboundedness of $\int \mathrm{e}^{2B}/(\int \mathrm{e}^{B})^2$, which we observed in Section 3.

(2) We can rewrite Corollary 5.1 as

$$\int \exp\left(yq\frac{\int \mathrm{e}^{2B}}{(\int \mathrm{e}^{B})^2}\right) \mathrm{d}\nu^{0,0}_{(yc/2)^{-1}}(B) = \exp\left\{\frac{1}{yc}\left(\frac{x}{\sin(x)} - \frac{x}{\tan(x)}\right)\right\}, \quad (5.36)$$

where $x = \sqrt{q/c}$. Although possibly coincidental, it is worth noting that $x/\sin(x)$ and $x/\tan(x)$ are related to the $\hat{A}$-genus and the L-genus, respectively, which are in turn related to the $N = 1$ supersymmetry which we considered in Section 2, via elliptic cohomology (see [5,15]).

(3) To obtain orbital integrals corresponding to orbits above the universal Teichmuller orbit, it is necessary to regularize the divergent integrals above. One possible approach, which applies to the elliptic orbits, is simply to eliminate the up directions ($2(n-1)$-dimensional for Ell_n). But this breaks the $\mathcal{D}$-symmetry, and it is no longer clear how to evaluate the integrals.

We can also apply Corollary 5.1 to show that $\mathrm{d}\nu_T(b) \times \mathrm{d}\tau$ is asymptotically invariant in a restricted sense.

Corollary 5.2. *Abbreviate* $\mathrm{d}\nu_T(b) \times \mathrm{d}\tau$ *to* $\mathrm{d}\nu_T(\psi)$, *viewed as a probability measure on* $\mathcal{D}_{C^1}$. *Then for* $\phi \in \mathcal{D}$, *we have*

$$\int \left|\frac{\mathrm{d}\nu_T(\phi \circ \psi)}{\mathrm{d}\nu_T(\psi)} - 1\right|^p \mathrm{d}\nu_T(\psi) \to 0 \quad \text{as } T \uparrow \infty,$$

provided $p < \pi^2$.

Proof. In Section 4.2 of Chapter IV of [11] we observed that to prove this, using a dominated convergence argument, it suffices to prove that

$$\lim_{T\uparrow\infty} \int \exp\left(\frac{p}{2T}\frac{\int \mathrm{e}^{2b}}{(\int \mathrm{e}^{b})^2}\right) \mathrm{d}\nu_T(b) = 1, \quad (5.37)$$

see (3.2.19)–(3.2.21) of Chapter IV of [11]. This follows immediately from Corollary 5.1, with $c = 2$, $y = T^{-1}$, $q = p/2$. □

Appendix A. The orbit method and $\widetilde{\mathbf{PSU}(1,1)}$

In this appendix we briefly recapitulate some of the main points concerning the application of the orbit method to $\widetilde{\mathrm{PSU}(1,1)}$.

The Lie algebra is

$$\mathfrak{g} = su(1,1) = \left\{ \begin{pmatrix} \mathrm{i}E & \bar{p} \\ p & -\mathrm{i}E \end{pmatrix} : E \in \mathbb{R},\ p \in \mathbb{C} \right\}. \tag{A.1}$$

The Ad-invariant form $m^2 = E^2 - |p|^2$ (essentially the Killing form, which is nonexistent in the Virasoro case) allows us to identify Ad and Ad* (and $\mathfrak{g}$ with $\mathbb{R}^{1,2}$; hence m is rest mass, E is energy, and $p = p_1 + \mathrm{i}p_2$ is the momentum vector). The coadjoint orbits are then precisely the rest mass shells.

The irreducible unitary representations of $\widetilde{\mathrm{PSU}(1,1)}$ are well-known (see [12]). The highest (respectively, lowest) weight series corresponds to the shells with $m^2 > 0$ and $E > 0$ (respectively, $E < 0$), denoted $\mathcal{O}_m^{\pm}$. The principal series corresponds to the tachyonic shells with $m^2 < 0$, and the complementary series involves an exotic inner product for a nonunitary induced representation not clearly associated to any given orbit. We are mainly interested in the first series, but in the end we will see that all three interact for small values of the parameters, and in Fig. 3 we have attempted to convey the topology of the unitary dual.

Let $\Delta = \{w \in \mathbb{C} : |w| < 1\}$, and let κ denote the holomorphic cotangent bundle, equipped with the Poincare Hermitian metric $|(\mathrm{d}w)|_\kappa^2 = (1-|w|^2)^2$. The group PSU(1, 1) acts by automorphisms of κ (as a Hermitian, holomorphic line bundle), covering its action by linear fractional transformations of Δ, and

$$\omega = \mathrm{curv}(\kappa) = -\mathrm{i}\partial\bar{\partial}\log|(\mathrm{d}w)|^2 = 2\mathrm{i}\frac{\mathrm{d}w \wedge \mathrm{d}\bar{w}}{(1-|w|^2)^2} \tag{A.2}$$

is an invariant symplectic form. For any $m > 0$ there is a corresponding action of $\widetilde{\mathrm{PSU}(1,1)}$ on $\kappa^{m/2}$. The (moment) map

$$\mu : (\Delta, \mathrm{curv}(\kappa^{m/2})) \to \mathcal{O}_{m,+} : w \to (p, E) = \frac{m}{(1-|w|^2)}(2\mathrm{i}w, (1+|w|^2)) \tag{A.3}$$

is an equivariant isomorphism.

We now apply geometric quantization to this orbit. At this point we must make a choice. Do we map the orbit to holomorphic sections of the corresponding line bundle, or do we twist by half-densities? We will choose the first option (the ‘naive correspondence’). Either way we would encounter surprises at a later point. Thus we associate to the orbit $\mathcal{O}_{m,+}$ the

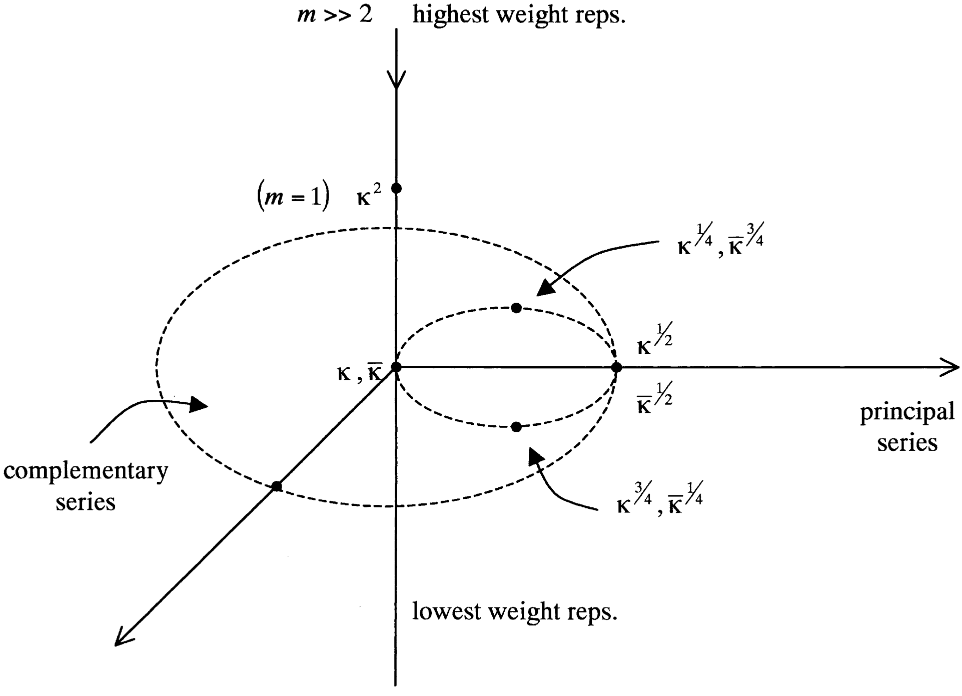

Fig. 3. Unitary dual of PSU(1, 1).

unitary representation

$$\widetilde{\mathrm{PSU}(1,1)} \times H^0_{L^2}(\kappa^{m/2}) \to H^0_{L^2}(\kappa^{m/2}) : \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}, f(z)(\mathrm{d}z)^{m/2}$$
$$\to f\left(\frac{-\bar{b}+az}{\bar{a}-bz}\right)\frac{1}{(\bar{a}-bz)^m}(\mathrm{d}z)^{m/2} \tag{A.4}$$

($H^0_{L^2}$ denotes holomorphic, L^2 sections) where

$$|f(\mathrm{d}z)^{m/2}|^2_{L^2} = \int_\Delta |f(z)(\mathrm{d}z)^{m/2}|^2_{\kappa^{m/2}}\omega = \int_\Delta |f(z)|^2(1-|z|^2)^{m-2}\,\mathrm{d}x\wedge\mathrm{d}y$$
$$= \sum_{n\geq 0}|f_n|^2 2\pi\int_0^1 r^{2n}(1-r^2)^{m-2}r\,\mathrm{d}r$$
$$= \frac{2\pi}{m-1}\sum_{n\geq 0}|f_n|^2\frac{\Gamma(n+1)\Gamma(m)}{\Gamma(m+n)}, \tag{A.5}$$

and $f=\sum f_n z^n$.

[Note: Our realization above does not explicitly indicate why this is, in general, necessarily a representation of the universal covering. To clarify this, it is convenient to realize

$\widetilde{\mathrm{PSU}(1,1)}$ as the set of pairs $\tilde{g} = (g, A) \in \mathrm{SU}(1,1) \times \mathbb{C}$ such that $\mathrm{e}^A = a$, $g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$, and the multiplication is given by

$$\tilde{g}_1 \tilde{g}_2 = \tilde{g}_3, \qquad A_3 = A_1 + A_2 + \log\left(1 + \frac{b_1 \bar{b}_2}{a_1 a_2}\right).$$

The precise interpretation of $(-\bar{b}z + a)^{2m}$ in (A.4) as a holomorphic function of $z \in \Delta$ is given by

$$(a - \bar{b}z)^{2m} = \mathrm{e}^{2mA}\left(1 - \frac{\bar{b}}{a}z\right)^{2m}, \tag{A.6}$$

which (for nonintegral m) does depend upon $\tilde{g}$, not merely g.]

As it stands this space is nonempty only if $m > 1$. However the “vacuum” $(\mathrm{d}z)^{m/2}$ has norm squared $2\pi/(m-1)$. It is therefore natural to rescale the inner product and define

$$\langle\langle f(\mathrm{d}z)^{m/2}, g(\mathrm{d}z)^{m/2}\rangle\rangle = \frac{m-1}{2\pi}\langle f(\mathrm{d}z)^{m/2}, g(\mathrm{d}z)^{m/2}\rangle_{L^2}. \tag{A.7}$$

From the last expression in (A.5), we see that this renormalized inner product can be analytically continued to $m > 0$, since the coefficients $B(n+1, m) = \Gamma(n+1)\Gamma(m)/\Gamma(m+n) > 0$.

For $m = 1$, i.e. for the representation corresponding to $\mathcal{O}_{1,+}$, there is a remarkable geometric interpretation of the rescaled inner product, as an integration process at the ideal boundary of the Poincare disk:

$$\lim_{m\downarrow 1}\frac{m-1}{2\pi}\langle f(\mathrm{d}z)^{m/2}, g(\mathrm{d}z)^{m/2}\rangle_{L^2} = \sum_{n\geq 0} f_n \bar{g}_n = \int_{S^1} f(\mathrm{d}z)^{1/2}\bar{g}(\mathrm{d}\bar{z})^{1/2}. \tag{A.8}$$

The point is that $\kappa^{1/2}$ extends to the boundary, and its restriction to S^1 is the same as the odd spin structure on S^1. The corresponding space of spinors, $\Omega^{1/2}_{\mathrm{odd}} = \Omega^0(\kappa|_{S^1})$, has a Hilbert space structure,

$$\Omega^{1/2}_{\mathrm{odd}} \otimes \Omega^{1/2}_{\mathrm{odd}} \to \mathbb{C} : \phi \otimes \psi \to \int_{S^1} \phi\bar{\psi}, \tag{A.9}$$

which is invariant with respect to the natural action of $\mathcal{D}^{(2)}$, the double cover. There is also a $\mathrm{SU}(1,1)$-invariant polarization

$$H_a = H_a^+ \oplus H_a^-, \tag{A.10}$$

where H_a^+ consists of boundary values of holomorphic sections of $\kappa^{1/2}$. The representation H_a^- is the dual of H_a^+, and it corresponds to $\mathcal{O}_{1,-}$.

Note that in the case $m = 0$, there is an intertwining operator

$$0 \to \mathbb{C} \to H^0(\Delta) \xrightarrow{\partial} H^1(\Delta) \to 0.$$

One of the puzzles of the orbit method, from our perspective, is that there does not seem to be any geometric interpretation of the inner product in the cases $0 < m < 1$ (the part of the highest weight series which does not contribute to the Plancherel formula). Note that these are not bizarre representations: For example undoubtedly the most important representation of $\widetilde{\mathrm{PSU}(1,1)}$ is the oscillator or metaplectic representation, which is the sum of the two irreducible representations corresponding to the orbits $\mathcal{O}_{1/2,+}$ and $\mathcal{O}_{3/2,+}$ (hence is actually a representation of $\mathrm{PSU}(1,1)^{(4)}$, the metaplectic group).

Each of the representations above, corresponding to the orbits $\mathcal{O}_{m,+}$, $m > 0$, has a holomorphic extension to the universal covering of the complex semigroup $\mathrm{PSL}(2,\mathbb{C})^+$ (consisting of linear fractional transformations mapping D into D^0), given by the same formula (A.5).

Now consider the other direction of the orbit correspondence, from the character point of view. By (A.5)

$$\begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} \cdot z^n (dz)^{m/2} = e^{i(n+(m/2))\theta} z^n (dz)^{m/2}. \tag{A.11}$$

If we analytically continue into the disk, i.e. we consider the action of the universal covering of $\mathbb{C}^*_{<1} \subset \mathrm{PSL}^+(2,\mathbb{C})$, then we obtain a regularization of the character

$$\chi_m(\tau) = \mathrm{tr}(q = e^{i2\pi\tau}) = \frac{e^{i\pi\tau m}}{1-q}, \tag{A.12}$$

where we have replaced $e^{i\theta}$ by $q = e^{i2\pi\tau}$, $\mathrm{Im}(\tau) > 0$.

Remark.

(a) (A.12) is the global expression for the holomorphic character, because the conjugacy classes of $\mathrm{PSL}(2,\mathbb{C})^+$ are parameterized by $|q| < 1$ (see 1.3 of [2]).
(b) Harish-Chandra developed another method of regularizing the trace, by initially viewing it as a distribution and eventually proving that the distribution is defined by a function on the regular set; when one takes the limit of the above formula at the boundary, one obtains Harish-Chandra's formulas for all the different components of the regular set (see 1.5 of [2]).

Now assume that $m > 1$. The Kirillov character formula is given by

$$\chi_m\left(\exp\left(\begin{pmatrix} x & \\ & -x \end{pmatrix}\right)\right) = j(x)^{-1} \int_{\mathcal{O}_{m-1,+}} e^{-i\left\langle \begin{pmatrix} x & \\ & -x \end{pmatrix}, \begin{pmatrix} iE & \bar{p} \\ p & -iE \end{pmatrix} \right\rangle} \omega_{\mathcal{O}_{m-1,+}}, \tag{A.13}$$

where $j(x) = \det^{1/2}\left(\sinh\left(\mathrm{ad}\begin{pmatrix} x/2 & \\ & -x/2 \end{pmatrix}\right)\right) / \mathrm{ad}\begin{pmatrix} x/2 & \\ & -x/2 \end{pmatrix} = \sinh(x)/x$, which reduces to the calculus identity

$$\frac{e^{mx}}{1-e^{2x}} j(x) = \int_\Delta e^{(m-1)x(1+|w|^2/1-|w|^2)} (1-|w|^2)^{-2}\, dw \wedge d\bar{w}. \tag{A.14}$$

Note that in the Kirillov formula, the orbit has been shifted by δ, half the sum of the positive roots. This calculation is valid only for $m > 1$. However (A.13) is also valid for $m = 1$, where $\mathcal{O}_{0,+}$ is the forward light-cone, which can be obtained by taking the limit $m \downarrow 1$ (or direct calculation).

An apparent paradox of the orbit method is that there does not seem to be room for an orbital interpretation of the characters that correspond to $0 < m < 1$ (and if we had used half-densities previously, when applying geometric quantization, we would not have attached these representations to any orbits at all). This phenomenon is closely related to the existence of the complementary series of unitary representations, and is perhaps related to our failure in the text to find orbital integral representations for the discrete series characters of the Virasoro algebra, which involve small values of the central charge.

As described in [12], one can parameterize the principal and complementary series by $\mathcal{C}_q^{(\tau)}$, where q is the value of the Casimir operator, the spectrum of the translation subgroup $\mathbb{R} \subset \widetilde{\mathrm{PSU}}(1,1)$ is $\exp(2\pi \mathrm{i}(\tau+\mathbb{Z}))$, where $0 \le \tau < 1$, and $\tau(1-\tau) < q$. For $m < 0$, the orbit $\mathcal{O}_m$ corresponds to the family of principal series representations $\mathcal{C}_q^{(\tau)}$, where $q = (1/4)+m$. The characters of these representations can be represented as orbital integrals (the parameter τ corresponds to the specification of a character for the stability subgroup of the orbit). The complementary series corresponds to the $\mathcal{C}_q^{(\tau)}$ with $q \le 1/4$; the corresponding characters apparently do not arise as orbital integrals. For any fixed τ, in the topology of the unitary dual, we have

$$\mathcal{C}_q^{(\tau)} \to D_\tau + \bar{D}_{1-\tau} \quad \text{as } q \downarrow \tau(1-\tau). \tag{A.15}$$

By mapping the (τ, q) strip to the plane minus the unit disk centered at $(\frac{1}{2}, 0)$, we obtain the picture shown in Fig. 3 of the unitary dual, which is intended to convey the topology.

In this picture the representations correspond to the points along the z-axis and the xy-plane minus the unit disk centered at $(1, 0)$. For $m > 2$ the highest (respectively, lowest) weight representations are along the upper (respectively, lower) z-axis. As m decreases from 2 to 0, the representations wind counterclockwise (respectively, clockwise) around the unit circle centered at $(1, 0)$, reflecting the limit (A.15). From a geometric point of view, the critical nature of κ and $\bar{\kappa}$ ($m = 2$) corresponds to the fact that the corresponding Hilbert space is *locally* conformally invariant. The geometry of the picture also reflects the critical nature of $\kappa^{1/4}$ and $\bar{\kappa}^{3/4}$ ($m = 1$).

Notably the picture does not suggest the critical nature of $\kappa^{1/2}$ and $\kappa^{3/2}$, which correspond to the metaplectic representation and the interface with the Heisenberg algebra. This is analogous to the apparent fact that the orbit picture for the Virasoro algebra does not hint at the existence of the discrete series unitary representations, which are related to an interface with affine algebras via the coset construction.

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